Some Additions to Review Set 2

1. What are the two ways we defined matrix multiplication, AB? (Hint: One way was the rwo-column rule).

   SOLUTION: $AB = A[\mathbf{b}_1 \quad \ldots \quad \mathbf{b}_k] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \ldots A\mathbf{b}_k]$

2. Let A, B be given below. Form the matrix product AB, if defined:

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{SOLUTION} \quad AB = \begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}$$

3. Given the matrix A, B below:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 3 & 2 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   (a) Compute only the $(2, 3)$ entry of AB: SOLN: -1

   (b) Compute only the $(3, 2)$ entry of AB^T: SOLN: 8

   (c) Compute $B - 3I_3$: SOLUTION:

$$\begin{bmatrix} -2 & -1 & 2 \\ 2 & -2 & 1 \\ 1 & 0 & -2 \end{bmatrix}$$

   (d) Compute C_{23} for matrix A (that's the (2,3) cofactor).

$$(-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = (-1)(-4) = 4$$

4. If A is the 2×3 matrix below, find a matrix C so that $AC = I$, but note that C is not the inverse of A. To simplify your computations, I've given you one form for C that you might use.

$$A = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ 0 & 0 \end{bmatrix}$$

   SOLUTION: Given the matrices above, we need to solve the systems below (the first one for c_{11} and c_{21} and the second for c_{12} and c_{22}.

$$\left[\begin{array}{rr|r} -1 & 2 & 1 \\ 6 & -9 & 0 \end{array}\right] \qquad \left[\begin{array}{rr|r} -1 & 2 & 0 \\ 6 & -9 & 1 \end{array}\right]$$

We can solve these either by matrix inversion or using Cramer's rule. The matrix C is given by

$$\begin{bmatrix} 3 & 2/3 \\ 2 & 1/3 \\ 0 & 0 \end{bmatrix}$$

5. Suppose A is $n \times n$ with the property that $A\mathbf{x} = \vec{0}$ has only the trivial solution. Without using the invertible matrix theorem, explain directly why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for every $\mathbf{b}$.

   SOLUTION: If $A\mathbf{x} = \vec{0}$ has only the trivial solution, then the row reduced echelon form of A has no non-pivot columns, therefore, we have n pivot columns, and therefore, all n rows are pivot rows. Thus, $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$.

6. Explain why the columns of A^2 span $\mathbb{R}^n$ whenever the columns of A are linearly independent. (Hint: You might think about whether or not A^2 must be invertible).

   SOLUTION: By the Invertible Matrix Theorem, if the columns of A are linearly independent (and A is square since A^2 is presumably defined), then A is invertible. If A is invertible, so is $A * A$, which is A^2. Since A^2 is invertible, its columns span $\mathbb{R}^n$.

7. Suppose subspace H is the span of the two vectors below in set $\mathcal{B}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\} = \{\mathbf{v}_1, \mathbf{v}_2\}$$

   (a) Does $\mathcal{B}$ span $\mathbb{R}^3$? Why or why not?
   SOLUTION: No. We would need three vectors to span $\mathbb{R}^3$.

   (b) Find $[\mathbf{v}_1]_{\mathcal{B}}$
   Since $\mathbf{v}_1 = 1 \cdot \mathbf{v}_1 + 0 \cdot \mathbf{v}_2$, the coordinates are $(1, 0)$.

   (c) If $\mathbf{c} = (3, 3, 0)$, find $[\mathbf{c}]_{\mathcal{B}}$
   SOLUTION: You should find that the coordinates are $(1, 1)$.