SVD Handout - Homework Solutions

1. Exercises 1, 2 p. 481: Find the singular values of the matrices below:

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} -5 & 0 \\ 0 & 0 \end{bmatrix}$$

   SOLUTION: Recall that the singular values are $\sigma_i = \sqrt{\lambda_i}$, where the $\lambda' s$ are the eigenvalues of either AA^T or A^TA (remember that the non-zero eigenvalues are the same for both). In each case, the matrix is symmetric meaning that $A^TA = A^2 = AA^T$:

$$A^TA = A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \qquad \Rightarrow \sigma_1 = 3, \sigma_2 = 1$$

   (They should be in order from largest to smallest)

$$A^TA = A^2 = \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix} \qquad \Rightarrow \sigma_1 = \sqrt{5}, \sigma_2 = 0$$

2. Exercises 7, 9 p. 481: Construct the SVD of each matrix below (by hand):

$$\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix}$$

   Side note: As a disclaimer, these are textbook problems. I would have made the numbers work out more nicely for problems we do by hand...

   SOLUTIONS: As an outline, we compute either A^TA or AA^T to start, then compute the eigenvalues and eigenvectors. From there, we can also compute the eigenvectors to the other matrix product. In these examples, I'll compute the expansion for A^TA first, but this is not necessary.

   - For the first matrix,

$$A^TA = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix} \qquad \lambda^2 - 13\lambda + 36 = 0 \qquad \lambda = 4, 9$$

     For $\lambda = 4$, we have

$$(A^TA - 4I) = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

     Similarly, for $\lambda = 9$, we have

$$(A^TA - 9I) = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Also, we can construct $\mathbf{u} = A\mathbf{v}$:

$$\mathbf{u} = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

These vectors haven't been scaled appropriately yet- Now we put it all together:

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \qquad V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \qquad U = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

- For the second matrix,

$$A^T A = \begin{bmatrix} 74 & 32 \\ 32 & 26 \end{bmatrix} \qquad \lambda^2 - 100\lambda + 900 = 0 \qquad \lambda = 10, 90$$

For $\lambda = 10$, we have

$$(A^T A - 10I) = \begin{bmatrix} 64 & 32 \\ 32 & 16 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Similarly, for $\lambda = 90$, we have

$$(A^T A - 90I) = \begin{bmatrix} -16 & 32 \\ 32 & -64 \end{bmatrix} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(It was a coincidence that this matrix V is the same as the previous one!) Also, we can construct $\mathbf{u} = A\mathbf{v}$:

$$\mathbf{u} = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ 0 \\ 5 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 7 & 1 \\ 0 & 0 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \\ 15 \end{bmatrix}$$

In this problem, for the full SVD, we'll also need the eigenspace for AA^T, where $\lambda = 0$:

$$AA^T = \begin{bmatrix} 50 & 0 & 40 \\ 0 & 0 & 0 \\ 40 & 0 & 50 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(Looking back, we could have guessed this!) These vectors haven't been scaled appropriately yet- Now we put it all together. Remember that Σ has the same size as A, so it is 3×2, U is 3×3 and V is 2×2.

$$\Sigma = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & \sqrt{10} \\ 0 & 0 \end{bmatrix} \qquad V = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \qquad U = \begin{bmatrix} 1/\sqrt{2} & -1\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

3. Suppose $A = U\Sigma V^T$ is the (full) SVD.

(a) Suppose A is square and invertible. Find the SVD of the inverse.

SOLUTION: $A^{-1} = V\Sigma^{-1}U^T$

(b) If A is square, show that $|\det(A)|$ is the product of the singular values.

SOLUTION: Take the determinant of the SVD product. Remember that in this case, all the matrices must be square for this to be defined.

$$\det(A) = \det(U\Sigma V^T) = \det(U)\det(\Sigma^{-1})\det(V^T)$$

Since U is an orthogonal matrix, then $UU^T = I$. This makes the determinant easy:

$$\det(U^TU) = (\det(U))^2 = 1 \quad \Rightarrow \quad \det(U) = \pm 1$$

Similarly, $\det(V) = \pm 1$. Finally, the determinant of Σ is the product of the diagonal elements (since Σ is a diagonal matrix). Therefore,

$$\det(A) = \pm\sigma_1\sigma_2\cdots\sigma_n$$

so that the absolute value of the determinant is the product of the singular values.

(c) If A itself is symmetric, show that $U = V$ so that the SVD gives the eigenvalue-eigenvector factorization: PDP^T.

This actually comes from the Spectral Theorem.

4. Let A, $\mathbf{b}$ be as defined below. Use the SVD to write $\mathbf{b} = \hat{\mathbf{b}} + \mathbf{z}$, where $\hat{\mathbf{b}} \in \mathrm{Col}(A)$ and $\mathbf{z} \in \mathrm{Null}(A)$.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Also write in Matlab how you would check to see if $\hat{\mathbf{b}}$ is actually in the column space of A (using the output of the SVD).

TYPO: Did you notice? $\mathbf{z}$ can't be in the null(A), it has to be in the null(A^T) since $\mathbf{b}$ is in $\mathbb{R}^4$.

SOLUTION: The idea is to use the orthonormal bases for the colum and null space of A that we get from the SVD. First, in Matlab, we'll compute the SVD (and because the columns of A are linearly independent, the rank is 2):

```
A=[1 2;3 4;1 1;1 -1]; b=[1;2;3;4];
[U,S,V]=svd(A);
b_hat=U(:,1:2)*U(:,1:2)'*b;
z=U(:,3:4)*U(:,3:4)'*b;
%Test:
b_hat+z
```

To check the projection, recall that the projecting the projection doesn't change the vector. So you could see if $\hat{B}$ is the same as `U(:,1:2)*U(:,1:2)'*b_hat`.

5. (Exercise 12, 6.5) Given the data below, and the model equation

$$\beta_0 + \beta_1 \ln(w) = p$$

form a linear systems of equations to find the constants β_0, β_1, and find them by computing the pseudoinverse (using an appropriate SVD).

w	48	61	81	113	131
p	91	98	103	110	112

SOLUTION:

```
w=[48;61;81;113;131]; p=[91;98;103;110;112];
n=length(w);
A=[ones(n,1) log(w)];  %log is ln on Matlab
[U,S,V]=svd(A);
invS=[1/S(1,1),0; 0, 1/S(2,2)];
pinvA=V(:,1:2)*invS*U(:,1:2)';

beta=pinvA*p
```

6. Consider the matrix A below. We have two algorithms that will produce a basis for the column space- Gram-Schmidt and the SVD. Use Matlab to get both, then compare and contast the two bases. Hint: What vector comes first? Can we compare one, two or three dimensional subspaces?

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

SOLUTION: First, do the decompositions:

```
A=[1 3 5; 1 1 0; 1 1 2; 1 3 3];
%Gram-Schmidt on the columns of A comes from QR:
[Q,R]=qr(A);
[U,S,V]=svd(A);
%A has rank 3, so compare the three cols of Q with U
```

To compare subspaces, consider using the RREF of the matrix $[Q|U]$. If we do that, we get:

```
1    0    0    0     0.8663   -0.4082    0.2879         0
0    1    0    0    -0.4554   -0.4082    0.7912         0
0    0    1    0     0.2055    0.8165    0.5396         0
0    0    0    1    -0.0000   -0.0000         0    1.0000
```

You might notice that Q is 4×4, where if we computed it manually, it would be 4×3- You'll also note that the fourth columns of Q and U are the same- The single column vector is in the null(A^T). Therefore, the first three coumns of Q and U form different bases for the column space of A, but we see that they are the same space.

7. (Using the matrix A from the previous problem) Here is some data in $\mathbb{R}^4$ that we organize in an array

$$X = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Find the projection of this data into the space spanned by the first two columns of the matrix U from the SVD of A. It is OK to write only two significant digits if you're copying the output by hand.

SOLUTION: A bit of a typo, but this is recoverable if we consider this to be two vectors in $\mathbb{R}^4$ (so the matrix is transposed). The projection onto the space spanned by the first two columns of U is given by the following, where X has been entered as a 4×2 matrix rather than as a 2×4 matrix as given.

```
U(:,1:2)*U(:,1:2)'*X
```

8. We want to find a matrix P so that, given matrix A and vector $\mathbf{y}$, then $P\mathbf{y}$ is the projection of $\mathbf{y}$ into the column space of A (orthogonal projection).

We know that we could do this with the SVD, but using the normal equations and assuming that A is full rank, show that the matrix is:

$$P = A(A^TA)^{-1}A^T$$

(Hint: Start by writing $A\mathbf{x} = \mathbf{b}$ and get the normal equation.

SOLUTION:

$$A\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad A^TA\mathbf{x} = A^T\mathbf{b} \quad \Rightarrow \quad \mathbf{x} = (A^TA)^{-1}A^T\mathbf{b}$$

Now this is actually $\hat{\mathbf{x}}$, and we know that $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$. Therefore, multiply both sides of your equation by A:

$$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^TA)^{-1}A^T\mathbf{b} = P\mathbf{b}$$

9. Work through the movie example from the text- Be sure to run it in Matlab.