

## Solutions to the Complex Number Exercises

1. Suppose the roots to a cubic polynomial are  $a = 3$ ,  $b = 1 - 2i$  and  $c = 1 + 2i$ . Compute  $(x - a)(x - b)(x - c)$ .

SOLUTION: You could multiply it all out at once. If you take  $(x - b)(x - c)$  first, you get:

$$(x - a)(x - b)(x - c) = (x - 3)(x^2 - 2x + 5) = x^3 - 5x^2 + 11x - 15$$

2. Find the roots to  $x^2 - 2x + 10$ . Write them in polar form.

SOLUTION:

$$x = \frac{2 \pm \sqrt{2^2 - 4(10)}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

3. Show that:

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \quad \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

SOLUTION: If  $z = a + bi$ , then  $\bar{z} = a - bi$ , so:

$$z + \bar{z} = a + ib + (a - ib) = 2a = 2\operatorname{Re}(z)$$

And similarly,

$$z - \bar{z} = a + ib - (a - ib) = 2ib \quad \Rightarrow \quad \operatorname{Im}(z) = b = \frac{z - \bar{z}}{2i}$$

4. For the following, let  $z_1 = -3 + 2i$ ,  $z_2 = -4i$

- (a) Compute  $z_1\bar{z}_2$ ,  $z_2/z_1$

SOLUTION:

$$\begin{aligned} z_1\bar{z}_2 &= (-3 + 2i)(4i) = (8i^2) - 12i = -8 - 12i \\ \frac{z_2}{z_1} &= \frac{-4i}{-3 + 2i} \cdot \frac{-3 - 2i}{-3 - 2i} = \frac{-8 + 12i}{3^2 + 4^2} = -\frac{8}{13} + \frac{12}{13}i \end{aligned}$$

- (b) Write  $z_1$  and  $z_2$  in polar form.

SOLUTION: For  $z_1$ , the point  $(-3, 2)$  is in Quadrant II. To find the argument (angle) for  $z_1$ , we have to add  $\pi$  to the arctangent:

$$\theta = \tan^{-1}\left(\frac{2}{-3}\right) + \pi \approx 2.554 \text{ rad}$$

The length is:  $r = \sqrt{3^2 + 2^2} = \sqrt{13}$ .

We can verify this, by checking that:

$$\sqrt{13}\cos(\theta) \approx -3 \quad \sqrt{13}\sin(\theta) \approx 2$$

Writing the answer in polar form,  $re^{i\theta}$ ,

$$-3 + 2i = \sqrt{13}e^{2.554i}$$

For  $z_2$ , it is much simpler:  $r = 4$  and  $\theta = -\pi/2$ . Therefore,

$$-4i = re^{i\theta} = 4e^{-i\pi/2}$$

5. In each problem, rewrite each of the following in the form  $a + bi$ :

- (a)  $e^{1+2i}$

$$e(\cos(2) + i\sin(2))$$

- (b)  $e^{2-3i}$

$$e^2(\cos(-3) + i\sin(-3)) = e^2(\cos(3) - i\sin(3))$$

(c)  $e^{i\pi}$

$$\cos(\pi) + i \sin(\pi) = -1$$

(d)  $2^{1-i}$  First, notice that  $2^{1-i} = 2 \cdot 2^{-i}$ , so we'll compute  $2^{-i}$  below:

$$2^{-i} = e^{\ln(2^{-i})} = e^{-i \ln(2)} = \cos(-\ln(2)) + i \sin(-\ln(2)) = \cos(\ln(2)) - i \sin(\ln(2))$$

Therefore,  $2^{1-i}$  can be expressed as:

$$2(\cos(\ln(2)) - i \sin(\ln(2)))$$

(e)  $e^{2-\frac{\pi}{2}i}$

$$e^2 (\cos(-\pi/2) + i \sin(-\pi/2)) = -ie^2$$

(f)  $\pi^i$

$$\pi^i = e^{\ln(\pi^i)} = e^{i \ln(\pi)} = \cos(\ln(\pi)) + i \sin(\ln(\pi))$$