Linear Algebra- Final Exam Review

1. Show that $\text{Row}(A) \perp \text{Null}(A)$.

   SOLUTION: We can write matrix-vector multiplication in terms of the *rows* of the matrix A. If A is $m \times n$, then:

$$A\mathbf{x} = \begin{bmatrix} \text{Row}_1(A) \\ \text{Row}_2(A) \\ \vdots \\ \text{Row}_m(A) \end{bmatrix} \mathbf{x} = \begin{bmatrix} \text{Row}_1(A)\mathbf{x} \\ \text{Row}_2(A)\mathbf{x} \\ \vdots \\ \text{Row}_m(A)\mathbf{x} \end{bmatrix}$$

   Each of these products is the "dot product" of a row of A with the vector $\mathbf{x}$.

   To show the desired result, let $\mathbf{x} \in \text{Null}(A)$. Then each of the products shown in the equation above must be zero, since $A\mathbf{x} = \mathbf{0}$, so that $\mathbf{x}$ is orthogonal to each row of A. Since the rows form a spanning set for the row space, $\mathbf{x}$ is orthogonal to every vector in the row space.

2. Let A be invertible. Show that, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, so are $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$. NOTE: It should be clear from your answer that you know the definition.

   SOLUTION: We need to show that the only solution to:

$$c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + c_3A\mathbf{v}_3 = 0$$

   is the trivial solution. Factoring out the matrix A,

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = 0$$

   Think of the form $A\hat{\mathbf{x}} = \mathbf{0}$. Since A is invertible, the only solution to this is $\hat{\mathbf{x}} = 0$, which implies that the only solution to the equation above is the solution to

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$$

   Which is (only) the trivial solution, since the vectors are linearly independent. (NOTE: Notice that if the original vectors had been linearly dependent, this last equation would have non-trivial solutions).

3. Find the line of best first for the data:

x	0	1	2	3
y	1	1	2	2

   Let A be the matrix formed by a column from $\mathbf{x}$ column of ones, then we form the normal equations $A^TA\mathbf{c} = A^T\mathbf{y}$ and solve:

$$A^TA\mathbf{c} = A^T\mathbf{y} \qquad \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \hat{\mathbf{c}} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

   The solution is $\hat{\mathbf{c}} = (A^TA)^{-1}A^T\mathbf{y} = \frac{1}{10}[4, 9]^T$, so the slope is 2/5 and the intercept is 9/10.

4. Let $A = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$. (a) Is A orthogonally diagonalizable? If so, orthogonally diagonalize it! (b) Find the SVD of A.

SOLUTION: For part (a), the matrix A is symmetric, so it is orthogonally diagonalizable. It is also a diagonal matrix, so the eigenvalues are $\lambda = -3$ and $\lambda = 0$. The eigenvalues are the standard basis vectors, so $P = I$, and $D = A$.

For the SVD, the eigenvalues of $A^T A$ are 9 and 0, so the singular values are 3 and 0. The column space is spanned by $[1, 0]^T$, as is the row space. We also see that

$$A\mathbf{v}_1 = \sigma_1 \mathbf{u}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

This brings up a good point- You may use either:

$$U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

or the reverse:

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

This problem with the ± 1 is something we don't run into with the usual diagonalization.

5. Let V be the vector space spanned by the functions on the interval $[-1, 1]$.

$$\{1, t, t^2\}$$

Use Gram-Schmidt to find an orthonormal basis, if we define the inner product:

$$\langle f(t), g(t) \rangle = \int_{-1}^{1} 2f(t)g(t)\, dt$$

SOLUTION: Let $\mathbf{v}_1 = 1$ (which is not normalized- We'll normalize later). Then

$$\mathbf{v}_2 = t - \mathrm{Proj}_{\mathbf{v}_1}(t) = t - \frac{\int_{-1}^{1} 2t\, dt}{\int_{-1}^{1} 2\, dt} 1 = t - 0 = t$$

$$\mathbf{v}_3 = t^2 - \mathrm{Proj}_{\mathbf{v}_1}(t^2) - \mathrm{Proj}_{\mathbf{v}_2}(t^2) = t^2 - \frac{\int_{-1}^{1} 2t^2\, dt}{\int_{-1}^{1} 2\, dt} 1 - \frac{\int_{-1}^{1} 2t^3\, dt}{\int_{-1}^{1} 2t^2\, dt} t$$

We note that the integral of any odd function will be zero, so that last term drops:

$$\mathbf{v}_3 = t^2 - \frac{4/3}{4} 1 = t^2 - \frac{1}{3}$$

6. Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be orthonormal. If

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

then show that $\|\mathbf{x}\|^2 = |c_1|^2 + \cdots + |c_p|^2$. (Hint: Write the norm squared as the dot product).

SOLUTION: Compute $\mathbf{x} \cdot \mathbf{x}$, and use the property that the vectors $\mathbf{v}_i$ are orthonormal:

$$(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) =$$

Since all dot products of the form $c_i c_k \mathbf{v}_i \cdot \mathbf{v}_k = 0$ for $i \neq k$, then the dot product simplifies to:

$$c_1^2\mathbf{v}_1 \cdot \mathbf{v}_1 + c_2^2\mathbf{v}_2 \cdot \mathbf{v}_2 + \cdots c_p^2\mathbf{v}_p \cdot \mathbf{v}_p$$

And since the vectors are normalized, this gives the result:

$$\|\mathbf{x}\|^2 = c_1^2 + \cdots + c_p^2$$

(Don't need the magnitudes here, since we're working with real numbers).

7. Short answer:

   (a) If $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}, v$ are orthogonal.

   SOLUTION: True- This is the Pythagorean Theorem.

   (b) Let H be the subset of vectors in $\mathbb{R}^3$ consisting of those vectors whose first element is the sum of the second and third elements. Is H a subspace?

   SOLUTION: One way of showing that a subset is a subspace is to show that the subspace can be represented by the span of some set of vectors. In this case,

$$\begin{bmatrix} a+b \\ a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

   Because H is the span of the given vectors, it is a subspace.

   (c) Explain why the image of a linear transformation $T : V \to W$ is a subspace of W

   SOLUTION: Maybe "Prove" would have been better than "Explain", since we want to go through the three parts:

      i. $0 \in T(V)$ since $0 \in V$ and $T(0) = 0$.

      ii. Let u, v be in $T(V)$. Then there is an x, y in V so that $T(x) = u$ and $T(y) = v$. Since V is a subspace, $x + y \in V$, and therefore $T(x+y) = T(x) + T(y) = u + v$ so that $u + v \in T(V)$.

      iii. Let $u \in T(V)$. Show that $cu \in T(V)$ for all scalars c. If $u \in T(V)$, there is an x in V so that $T(x) = u$. Since V is a subspace, $cu \in V$, and $T(cu) \in T(V)$. By linearity, this means $cT(u) \in T(V)$.

   (OK, that probably should not have been in the short answer section)

(d) Is the following matrix diagonalizable? Explain. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$

SOLUTION: Yes. The eigenvalues are all distinct, so the corresponding eigenvectors are linearly independent.

(e) If the column space of an 8×4 matrix A is 3 dimensional, give the dimensions of the other three fundamental subspaces. Given these numbers, is it possible that the mapping $\mathbf{x} \to A\mathbf{x}$ is one to one? onto?

SOLUTION: If the column space is 3-d, so is the row space. Therefore the null space (as a subspace of $\mathbb{R}^4$) is 1 dimensional and the null space of A^T is 5 dimensional. Since the null space has more than the zero vector, $A\mathbf{x} = \mathbf{0}$ has non-trivial solutions, so the matrix mapping will not be 1-1. Since the column space is a three dimensional subspace of $\mathbb{R}^8$, the mapping cannot be onto.

(f) i. Suppose matrix Q has orthonormal columns. Must $Q^TQ = I$?
SOLUTION: Yes, $Q^TQ = I$.

ii. True or False: If Q is $m \times n$ with $m > n$, then $QQ^T = I$.
SOLUTION: False- If $m \neq n$, then QQ^T is the projection matrix that takes a vector $\mathbf{x}$ and projects it to the column space of Q.

iii. Suppose Q is an orthogonal matrix. Prove that $\det(Q) = \pm 1$.
SOLUTION: If Q is orthogonal, then $Q^TQ = I$, and if we take determinants of both sides, we get:

$$(\det(Q))^2 = 1$$

Therefore, the determinant of Q is ± 1.

8. Find a basis for the null space, row space and column space of A, if $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

The basis for the column space is the set containing the first and third columns of A. A basis for the row space is the set of vectors $[1, 1, 0, 0]^T, [0, 0, 1, 1]^T$. A basis for the null space of A is $[-1, 1, 0, 0]^T, [0, 0, -1, 1]^T$.

9. Find an orthonormal basis for $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ using Gram-Schmidt (you might wait until the very end to normalize all vectors at once):

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

SOLUTION: Using Gram Schmidt (before normalization, which is OK if doing by hand), we get

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

10. Let $\mathbb{P}_n$ be the vector space of polynomials of degree n or less. Let W_1 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(0)\mathbf{p}(1) = 0$. Let W_2 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(2) = 0$. Which of the two is a subspace of $\mathbb{P}_n$?

    SOLUTION: We check the properties-

    - Is 0 in the subspace?

      For both W_1 and W_2, the zero polynomial satisfies both.

    - Is the subspace closed under addition?

      For W_1, let $p(t)$ be a polynomial such that $p(0)p(1) = 0$, and let $h(t)$ be another polynomial with that property, $h(0)h(1) = 0$.

      Does that imply that $g(t) = p(t) + h(t)$ has the desired property?

      $$g(0)g(1) = (p(0)+h(0))(p(1)+h(1)) = p(0)p(1)+p(0)h(1)+h(0)p(1)+h(0)h(1) = p(0)h(1)+h(0)p(1)$$

      For example, if $h(0) = 1$, $h(1) = 0$, $p(0) = 0$, $p(1) = 1$, then this quantity is not zero. Therefore, W_1 is not closed under addition.

      Alternate explanation: You can show that it doesn't work by providing a specific example: $p(t) = t$ has the property, and $h(t) = 1 - t$ also has the property (since it is zero at $t = 1$). When you add them, $g(t) = t + (1 - t) = 1$, which does not have the property (it is never zero).

      Going back to the rest: We can check that W_2 is closed under addition: Let $p(t), h(t)$ be two functions in W_2. Then $g(t) = p(t) + h(t)$ satisfies the property that

      $$g(2) = p(2) + h(2) = 0 + 0 = 0$$

    - Similarly, W_2 is closed under scalar multiplication- If $g(t) = cp(t)$, then $g(2) = cp(2) = c \cdot 0 = 0$.

    Therefore, W_2 is a subspace, and W_1 is not.

11. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

    $$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

    SOLUTION: For matrix A, $\lambda = 3, 5$. Eigenvectors are $[-1, 2]^T$ and $[-1, 1]^T$, respectively.

    For matrix B, for $\lambda = 3+i$, an eigenvector is $[1, i]^T$. The other eigenvalue and eigenvector are the complex conjugates.

    For matrix C, expand along the 2d row. $\lambda = 2$ is a double eigenvalue with eigenvectors $[0, 1, 0]^T$ and $[1, 0, 1]^T$. The third eigenvalue is $\lambda = 0$ with eigenvector $[-1, 0, 1]^T$.

12. Define $T : P_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

(a) Find the image under T of $p(t) = 5 + 3t$.
SOLUTION: $[2, 5, 8]^T$

(b) Show that T is a linear transformation.
SOLUTION: We show it using the definition.

i. Show that $T(p + q) = T(p) + T(q)$:

$$T(p+q) = \begin{bmatrix} p(-1) + q(-1) \\ p(0) + q(0) \\ p(1) + q(1) \end{bmatrix} = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} + \begin{bmatrix} q(-1) \\ q(0) \\ q(1) \end{bmatrix} = T(p) + T(q)$$

ii. Show that $T(cp) = cT(p)$ for all scalars c.

$$T(cp) = \begin{bmatrix} cp(-1) \\ cp(0) \\ cp(1) \end{bmatrix} = c \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix} = cT(p)$$

(c) Find the kernel of T. Does your answer imply that T is $1 - 1$? Onto? (Review the meaning of these words: kernel, one-to-one, onto)
SOLUTION:
Since the kernel is the set of elements in the domain that map to zero, let's see what what the action of T is on an arbitrary polynomial. An arbitrary vector in P_2 is: $p(t) = at^2 + bt + c$, and:

$$T(at^2 + bt + c) = \begin{bmatrix} a - b + c \\ c \\ a + b + c \end{bmatrix}$$

For this to be the zero vector, $c = 0$. Then $a - b = 0$ and $a + b = 0$, so $a = 0, b = 0$. Therefore, the only vector mapped to zero is the zero vector.
Side Remark: Recall that for any linear function T, if we are solving $T(x) = y$, then the solution can be written as $x = x_p + x_h$, where x_p is the particular solution (it solves $T(x_p) = y$), and $T(x_h) = 0$ (we said x_h is the homogeneous part of the solution). So the equation $T(x) = y$ has at most one solution iff the kernel is only the zero vector (if T was realized as a matrix, we get our familiar setting).

Therefore, T is $1 - 1$. The mapping T will also be onto (see the next part).

13. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ so that $\|\mathbf{v}\| = 1$, and let $Q = I - 2\mathbf{v}\mathbf{v}^T$. Show (by direct computation) that $Q^2 = I$.

SOLUTION: This problem is to practice matrix algebra:

$$Q^2 = (I - 2\mathbf{v}\mathbf{v}^T)(I - 2\mathbf{v}\mathbf{v}^T) = I^2 - 2I\mathbf{v}\mathbf{v}^T - 2\mathbf{v}\mathbf{v}^T I + 4\mathbf{v}\mathbf{v}^T\mathbf{v}\mathbf{v}^T = I - 4\mathbf{v}\mathbf{v}^T + 4\mathbf{v}(1)\mathbf{v}^T = I$$

14. Let A be $m \times n$ and suppose there is a matrix C so that $AC = I_m$. Show that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$. Hint: Consider $AC\mathbf{b}$.

    SOLUTION: Using the hint, we see that $AC\mathbf{b} = \mathbf{b}$. Therefore, given an arbitrary vector $\mathbf{b}$, the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = C\mathbf{b}$.

15. If B has linearly dependent columns, show that AB has linearly dependent columns. Hint: Consider the null space.

    SOLUTION: If B has linearly dependent columns, then the equation $B\mathbf{x} = \mathbf{0}$ has non-trivial solutions. Therefore, the equation $AB\mathbf{x} = \mathbf{0}$ has (the same) non-trivial solutions, and the columns of AB must be linearly dependent.

16. If λ is an eigenvalue of A, then show that it is an eigenvalue of A^T.

    SOLUTION: Use the properties of determinants. Given

$$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I^T| = |A^T - \lambda I|$$

    the solutions to $|A - \lambda I| = 0$ and $|A^T - \lambda I| = 0$ are exactly the same.

17. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelogram with vertices at $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{u} + v$. Compute the area of S.

    SOLUTION: The area of the parallelogram formed by two vectors in $\mathbb{R}^2$ is the determinant of the matrix whose columns are those vectors. In this case, that would be 4.

18. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.

    If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

    SOLUTION: This question reviews the relationship between the determinant and row operations. The determinant of B is 5. The determinant of C is -10. The determinant of BC is -50.

19. Let $1, t$ be two vectors in $C[-1,1]$. Find the length between the two vectors and the cosine of the angle between them using the standard inner product (the integral). Find the orthogonal projection of t^2 onto the set spanned by $\{1, t\}$.

    SOLUTION:

    - The length between the vectors is:

$$\sqrt{\langle (1-t), (1-t) \rangle} = \sqrt{\int_{-1}^{1} (1-t)^2 \, dt} = \sqrt{\frac{-1}{3}(1-t)^3 \Big|_{-1}^{1}} = \sqrt{\frac{8}{3}}$$

20. Define an *isomorphism:* A one-to-one and onto linear transformation between vector spaces (see p. 251)

    NOTE: An isomorphism was the critical piece to understanding when two vector spaces had the same "form"- For example, a plane through the origin in $\mathbb{R}^3$ and the plane $\mathbb{R}^2$ are not equal, but they are isomorphic; the isomorphism takes a point of the plane and returns its coordinates- That is, the plane in $\mathbb{R}^3$ is the span of two vectors in $\mathbb{R}^3$, so every point on the plane is a linear combination of those two. The point in $\mathbb{R}^2$ that we refer to is the ordered pair of weights from the linear combination.

    As another example, if the plan is spanned by $\mathbf{u}$ and $\mathbf{v}$ in vector space V, and $\mathbf{x}$ is on the plane so that $\mathbf{x} = c_1\mathbf{u} + c_2\mathbf{v}$, then the isomorphism takes $\mathbf{x} \in V$ and gives $(c_1, c_2) \in \mathbb{R}^2$.

21. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$

    Find at least two $\mathcal{B}-$coordinate vectors for $\mathbf{x} = [1, 1]^T$.

    SOLUTION: Row reduce to find $\mathbf{x}$ as a linear combination of the vectors in $\mathcal{B}$:

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -3 & -8 & 7 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 & 5 \\ 1 & 1 & 1 & -2 \end{bmatrix}$$

    If we label the weights as c_1, c_2 and c_3, then

$$\begin{aligned} c_1 &= 5 + 5t \\ c_2 &= -2 - t \\ c_3 &= t \end{aligned}$$

    Therefore, we could form $\mathbf{x}$ using the weights $(5, -2, 0)$, or $(10, -3, 1)$ or any combination given.

    NOTE: The columns do NOT form a basis for $\mathbb{R}^2$, but they do form a spanning set for $\mathbb{R}^2$. If the columns formed a basis, the weights for the linear combination would be unique (no free variables), but in this case, the expansion of $\mathbf{x}$ in this basis was not unique.

22. Let U, V be orthogonal matrices. Show that UV is an orthogonal matrix.

    SOLUTION: This question deals with the *definition* of an orthogonal matrix: A square matrix such that $U^T = U^{-1}$. First, if the product UV is defined, then U and V are both $n \times n$ for some n.

    Secondly, since U, V are each invertible, then so is UV. Furthermore,

$$(UV)^{-1} = V^{-1}U^{-1} = V^TU^T = (UV)^T$$

    Therefore, UV is an orthogonal matrix.

23. In terms of the four fundamental subspaces for a matrix A, what does it mean to say that:

- $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

  For this to be true, we know that $\mathbf{b} \in \text{Col}(A)$ (to be consistent), and that $\text{Null}(A) = \{\mathbf{0}\}$ (for the solution to be unique).

- $A\mathbf{x} = \mathbf{b}$ has no solution.

  For this to be true, $\mathbf{b}$ cannot be an element of the column space of A.

- In the previous case, what is the "least squares" solution? What quantity is being minimized?

  The least squares solution is the vector $\hat{\mathbf{x}}$ where the magnitude of the difference between the given $\mathbf{b}$ and $\hat{\mathbf{b}} = A\hat{\mathbf{x}}]$ is as small as possible. Therefore, we are minimizing the following, over all vectors $\mathbf{x}$:

  $$\|\mathbf{b} - A\mathbf{x}\|$$

- $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions.

  For the system to be consistent, $\mathbf{b} \in \text{Col}(A)$. For us to have an infinite number of solutions, the dimension of the null space is greater than 0 (or, the dimension of the null space is 1 or more).

24. Let T be a one-to-one linear transformation for a vector space V into $\mathbb{R}^n$. Show that for $\mathbf{u}, \mathbf{v}$ in V, the formula:

$$\langle u, v \rangle = T(\mathbf{u}) \cdot T(\mathbf{v})$$

defines an inner product on V.

SOLUTION: This was a homework problem from 6.7. We want to check the properties of the inner product, which are: (i) Symmetry, (ii) and (iii) Linear in the first coordinate, and (iv) Inner product of a vector with itself is non-negative (and the special case of 0).

(a) $\langle \mathbf{u}, \mathbf{v} \rangle = T(\mathbf{u}) \cdot T(\mathbf{v}) = T(\mathbf{v}) \cdot T(\mathbf{u}) = \langle \mathbf{u}, \mathbf{v} \rangle$, where the second equality is true because the regular dot product is symmetric.

(b)
$$\langle \mathbf{u} + \mathbf{w}, \mathbf{v} \rangle = T(\mathbf{u} + \mathbf{w}) \cdot T(\mathbf{v}) = (T(\mathbf{u}) + T(\mathbf{w})) \cdot T(\mathbf{v}) = T(\mathbf{u}) \cdot T(\mathbf{v}) + T(\mathbf{w}) \cdot T(\mathbf{v})$$

(c)
$$\langle c\mathbf{u}, \mathbf{v} \rangle = T(c\mathbf{u}) \cdot T(\mathbf{v}) = cT(\mathbf{u}) \cdot T(\mathbf{v})$$

(d)
$$\langle \mathbf{u}, \mathbf{u} \rangle = T(\mathbf{u}) \cdot T(\mathbf{u}) = \|T(\mathbf{u})\|^2$$

The dot product of a vector with itself is always non-negative. Furthermore, by the same equation, if $\langle u, u \rangle = 0$, then u must be the zero vector.

25. Describe all least squares solutions to $\begin{aligned} x + y &= 2 \\ x + y &= 4 \end{aligned}$

SOLUTION: Interesting to think about- In the plane, these are two parallel lines (each has a slope of -1, one has an intercept at 2, the other at 4).

Using linear algebra, we have

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad \Leftrightarrow \qquad A\mathbf{x} = \mathbf{b}$$

We cannot use the normal equations, because A does not have full rank (a rank of 2). However, if we project $\mathbf{b}$ into the column space of A (which is the span of $[1,1]^T$), then we can solve the sytem (and in fact, we'll have an infinite number of solutions since there will be a free variable):

$$\mathrm{Proj}_{\mathrm{Col}(A)}(\mathbf{b}) = \frac{2+4}{1+1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Now solve the system $A\hat{\mathbf{x}} = [3,3]^T$, which is:

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

If we let the free variable be $\hat{y} = t$, then $\hat{x} = 3 - t$. Notice that this set of points represents the line $\hat{y} = -\hat{x} + 3$, which is the line right down the middle between the other two lines!

26. Let $\mathbf{u} = [5,-6,7]^T$. Let W be the set of all vectors orthogonal to $\mathbf{u}$. (i) Geometrically, what is W? (ii) Find the projection of $\mathbf{x} = [1,2,3]^T$ onto W. (iii) Find the distance from the vector $\mathbf{x} = [1,2,3]^T$ to the subspace W.

    SOLUTIONS:

    - W is the plane in $\mathbb{R}^3$ going through the origin whose normal vector (in the sense of Calc 3) is $\mathbf{u}$.
    - We can write $\mathbf{x} = \hat{\mathbf{x}} + \mathbf{z}$, where $\hat{\mathbf{x}}$ is the projection onto $\mathbf{u}$, then $\mathbf{z}$ will be the desired vector in W:

    $$\hat{\mathbf{x}} = \left(\frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \frac{5-12+21}{25+36+49} \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix} = \begin{bmatrix} 7/11 \\ -42/55 \\ 49/55 \end{bmatrix} \approx \begin{bmatrix} 0.6364 \\ -0.7636 \\ 0.8909 \end{bmatrix} \quad \Rightarrow \quad \mathbf{z} \approx \begin{bmatrix} 0.36 \\ 2.76 \\ 2.11 \end{bmatrix}$$

    - The distance is then $\|\mathbf{x} - \mathbf{z}\| = \|\hat{\mathbf{x}}\| \approx 1.33$

27. The SVD can be used to determine whether a matrix is invertible, and can provide a formula for the inverse. The matrix A is invertible if it is square and all singular values are positive (not zero). Then the formula for the inverse is much the same as the formula for the pseudoinverse.