Dynamical System

- A set of points and a rule.
- For function iteration, $x_{i+1} = f(x_i)$
- Starting with x_0, we have:

$$x_0 \qquad x_1 = f(x_0) \qquad x_2 = f(f(x_0)) \qquad x_3 = f(f(f(x_0)))$$

  and so on...
- $x_0, x_1, x_2, \cdots$ is called the orbit of x_0.
- Question: What is the long term behavior of the orbit?

$$\lim_{t\to\infty} x_t = \lim_{t\to\infty} f^{(t)}(x_0)$$

Example 1

Let $x_0 = \frac{1}{2}$, and $f(x) = x^2$. Then:

i	0	1	2	3	$\cdots$
x_i	1/2	1/4	1/16	1/256	$\cdots$

This orbit seems to converge to zero.

What will the orbit of $x_0 = 2$ do? (Diverge)

Let $x_0 = 0$ and $x_0 = 1$. Orbits don't change.
These are the **Equilibrium Solutions**:

$$x = f(x) \qquad x = x^2 \qquad x^2 - x = 0 \qquad x(x-1) = 0 \qquad x = 0, 1$$

Summary of the dynamics using $f(x) = x^2$ and the real line:

$$\text{If } 0 < x_0 < 1 \quad \text{then} \quad x_i \to 0 \text{ as } i \to \infty$$

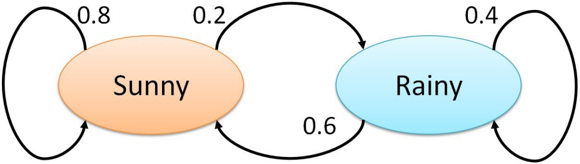
And

$$\text{If } x_0 > 1 \quad \text{then} \quad x_i \to \infty \text{ as } i \to \infty$$

The equilibrium $x_0 = 0$ is *stable*
The equilibrium $x_0 = 1$ is *unstable*

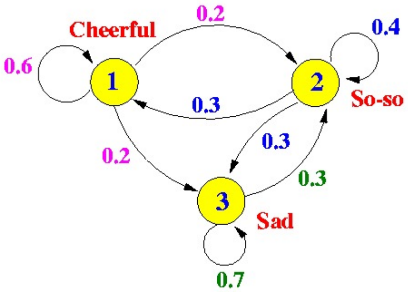
Example 2: Markov Chains

Markov chains can be used to simulate transitions between states using probabilities.



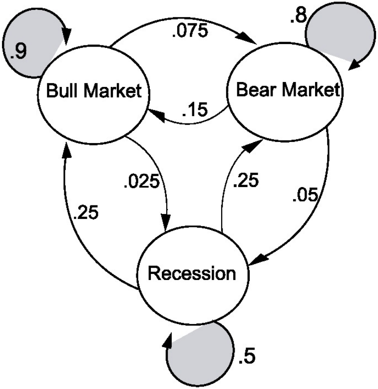
"If today sunny, the probability of sunny tomorrow is 80%, and the probability of rain is 20%".
"If rainy today, the probability of rainy tomorrow is 40%, and the probability of sun is 60%."

Using a random choice:
S S R R S R R R S R S S S R S R R S S S
Using a Markov Chain:
R S S S S S S S R S R R S S S S S R S S S



You can track your moods...

- C C - - C C C C C - S S - - C C C C C C



You might model the economy...

Markov Chain Simulated Conversation

First, get a large sample of writing (twitter conversations, Project Gutenberg).
Find transition probabilities.
Put the strings of words together!

Example:

- "I like" could be followed by "turtles" or "to" or "dogs.". Choose "to".
- "I like to" - "like to" could be followed by "know" or "toy" or "get". Choose "get".
- "I like to get"- "to get" could be followed by "down." or "rid" or "dressed" Choose "down."
- "down." has a period, which finishes phrase.

I like to get down.

Markov Chain Poetry

A “snowball” is a poem in which each line is a single word, each word is one letter longer.
(Search for “Nossidge snowball”- His code available online)

o
we
all
have
heard
people
believe
anything

Markov Chain Poetry

i
am
the
dawn
light
before
anybody
expected
something
disorderly

Some definitions

- A vector with positive entries that sum to 1: *Probability Vector*
- A matrix M with columns that are probability vectors is a *stochastic matrix*
  Example:

  $M =$

		From 1	From 2	From 3
To:	1	0.7	0.3	0.1
To:	2	0.2	0.5	0.0
To:	3	0.1	0.2	0.9

  The 0.3 entry means there is a probability of 30% of going to state 1 from state 2.
- A stochastic matrix is *regular* if there is a k so that P^k has all non-negative values. (The example is regular, if you check P^2).

Markov Chain Dynamics

EXAMPLE: Define our state vector as:

$$\mathbf{x} = \left[\begin{array}{c} \text{City} \\ \text{Suburbs} \end{array} \right]$$

Define the transition matrix:

	From	
	City	Subs
To City	0.95	0.03
To Subs	0.05	0.97

The dynamics: $\mathbf{x}_k = M^k \mathbf{x}_0$.

If 60% of the population is in the city, 40% in the suburbs, we can compute the percentages for next year:

$$\mathbf{x}_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = M\mathbf{x}_0 = \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix} = M^2\mathbf{x}_0 = \begin{bmatrix} 0.565 \\ 0.435 \end{bmatrix}$$

and so on...

If the orbit does converge, it will converge to a fixed point.

Definition: x is fixed if $x = f(x)$.

In this context, $\mathbf{x}$ is fixed if

$$\mathbf{x} = M\mathbf{x} \qquad M\mathbf{x} - \mathbf{x} = \vec{0} \qquad (M - I)\mathbf{x} = \vec{0}$$

For example,

$$(M - I)\mathbf{x} = \vec{0} \quad \Rightarrow \quad \begin{bmatrix} 0.95 - 1 & 0.03 \\ 0.05 & 0.97 - 1 \end{bmatrix} \mathbf{x} = \vec{0}$$

$$\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \mathbf{x} = \vec{0}$$

Therefore, if $\mathbf{x} = [c, s]^T$, then:

$$-5c + 3s = 0 \quad \Rightarrow \quad \begin{array}{ll} c & = 3/5s \\ s & = s \end{array}$$

$$= s \begin{bmatrix} 3/5 \\ 1 \end{bmatrix} = s_1 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = s_2 \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

If 3/8 of the population is in the city, 5/8 in the suburbs, then the populations will remain unchanged in time.

HOMEWORK: Exercises 1, 2, 3, 9, 11, 12