

Review Questions: Sections 1.1-2.1

The first exam will cover sections 1.1-1.5, 1.7-1.9, 2.1. You should be familiar with the homework. Typical exam-style questions are given below. There are many more questions here than will be on the exam, which should take about 50 minutes.

No calculators will be allowed on the exam.

Definitions and Basic Theorems

Definitions should be memorized, and you should be able to give the result of the basic theorems below, given their hypotheses as below.

1. Finish the definition:
 - (a) A **linear combination** of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is:
 - (b) A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ are said to be **linearly independent** if:
 - (c) The **span** of a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is:
 - (d) A system of equations is **inconsistent** if:
 - (e) A system of equations is **homogeneous** if:
 - (f) Two matrices are **row equivalent** if:
 - (g) A transformation $T : X \rightarrow Y$ is said to be **linear** if:
2. Given the **definition** of the operation shown:
 - Matrix-vector multiplication: $A\mathbf{x} =$
 - Matrix-matrix multiplication: $AB =$
3. What information about $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ do we need to know in order to compute the standard matrix for the transformation?
4. Fill in the blanks for the Existence and Uniqueness Theorem. Your answers should refer to pivot columns:
 - A linear system is consistent if and only if:
 - Furthermore, if the system is consistent, the solution is unique if:
5. Give three statements that are logically equivalent to saying that A has a pivot in every row. I'll give some hints so you can fill in the blanks:
 - For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ _____
 - _____ of A span \mathbb{R}^m
 - Each \mathbf{b} is a linear combination of _____ of A
 - The mapping $\mathbf{x} \rightarrow A\mathbf{x}$ will be _____ (choose from 1-1, onto, both or neither)

- Similar to the last problem, give two statements that are logically equivalent to saying that: A has a pivot in every column.
- Suppose A is $m \times n$ with $m > n$. Is it possible that the mapping $\mathbf{x} \rightarrow A\mathbf{x}$ is 1-1? Onto? (Explain).
- Same question, but let A be $m \times n$ with $m < n$.

Computational Questions

- Find the row reduced echelon form of the matrix A given below. Be sure to show all your work:

$$A = \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix}$$

- If the matrix given above was actually an augmented matrix, use your row reduced echelon form to give the solution to the system.
- Do the three lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$ and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.
- Let $\mathbf{a}_3 = 2\mathbf{a}_1 - 3\mathbf{a}_2$. Let $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$. If A is 3×3 with 2 pivots, write the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form.
- Write the equation of a plane that represents the span of the column vectors (written as rows to save space): $[1, 2, 3, 4]$, $[1, -1, -1, 1]$ and goes through the point $[3, 0, 0, 1]$.
- Find the general solution (in parametric vector form) to the system:

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= -1 \\ -2x_1 - 6x_2 - x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 + 3x_4 &= 2 \end{aligned}$$

- Suppose the solution set of a certain system of linear equations is given by $x_1 = 5 + 4x_4$, $x_2 = -2 + 7x_4$ with $x_3 = 2 + x_4$ and x_4 is a free variable.
 - Use vectors to describe the solution set as a (parametric) line in \mathbb{R}^4 .
 - Was the original system homogeneous? If not, give the solution to the homogeneous system of equations, if you have enough information.
- Show that the mapping T is not linear: $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 2x_2 \\ 3x_2 + 1 \end{bmatrix}$
- Given the matrix A below, explain whether or not the system $A\mathbf{x} = \mathbf{b}$ has a solution in terms of h . If there are restrictions on \mathbf{b} , give them.

$$A = \begin{bmatrix} 1 & -3 \\ 2 & -h \end{bmatrix}$$

10. Let A be a 3×4 matrix, let $\mathbf{y}_1, \mathbf{y}_2$ be vectors, and let $\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2$. Suppose that $\mathbf{y}_1 = A\mathbf{x}_1$ and $\mathbf{y}_2 = A\mathbf{x}_2$ for some vectors \mathbf{x}_1 and \mathbf{x}_2 .

- (a) What size must $\mathbf{y}_1, \mathbf{y}_2, \mathbf{x}_1, \mathbf{x}_2$ be?
(b) Does $A\mathbf{x} = \mathbf{w}$ have a solution? Why or why not?

11. Suppose that:

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$.

12. Let A and B be given below. (a) Find the second column of AB (without computing everything). (b) Find the $(3, 1)$ entry of $B^T A$.

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$$

13. Determine the matrix for the linear transformation T given below: $T(x_1, x_2, x_3, x_4) = 3x_1 - 4x_2 + 8x_4$
14. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$, where $T(\mathbf{e}_1) = (1, 4)$, $T(\mathbf{e}_2) = (-2, 9)$, and $T(\mathbf{e}_3) = (3, -8)$. Find a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$.
15. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ so that $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{bmatrix}$$

Is T 1-1? Explain. Is T onto? Explain.

16. Suppose we want to determine a quadratic function $f(x) = a_0 + a_1x + a_2x^2$ that interpolates the data $(1, 1)$, $(-1, 2)$, and $(2, 3)$. Write down the system of equations (and the corresponding matrix equation) we need to solve (do not actually solve the system) to find the polynomial.

Discussion Questions

These types of questions are more theoretical in nature. You do not have to refer to specific theorem numbers in your justifications, but you should note the existence of a theorem, if there is one to help. You should not argue “naively”, or from first principles- Use the material that we have learned.

1. Fill in the blank using the concept of pivot columns: “If a linear system is consistent, then the solution is unique if and only if _____”.
2. Suppose A is 3×3 and \mathbf{y} is a vector in \mathbb{R}^3 such that the equation $A\mathbf{x} = \mathbf{y}$ does not have a solution. Does there exist a vector \mathbf{z} in \mathbb{R}^3 such that $A\mathbf{x} = \mathbf{z}$ has a unique solution?

3. If $T : X \rightarrow Y$ and $S : Y \rightarrow Z$ are linear transformations, show that the composition is a linear transformation from X to Z .
4. Show that if the columns of B are linearly dependent, then so are the columns of AB .
5. Let A be $n \times n$. If the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, do the columns of A span \mathbb{R}^n ? Why or why not? Is your answer different if A is $n \times m$?
6. Let T be a linear transformation. Show that if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent vectors, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ are linearly dependent vectors.
7. If H is 7×7 matrix and $H\mathbf{x} = \mathbf{v}$ is consistent for every \mathbf{v} in \mathbb{R}^7 , then is it possible for $H\mathbf{x} = \mathbf{v}$ to have *more* than one solution for some $\mathbf{v} \in \mathbb{R}^7$? Why or why not?
8. Suppose that the third column of B is the sum of the first two columns. What can you say about the third column of AB ? Why?
9. Suppose that $CA = I_n$ (the $n \times n$ identity matrix). Show that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A cannot have more columns than rows.
10. If \mathbf{u}, \mathbf{v} are in \mathbb{R}^n , how are $\mathbf{u}^T \mathbf{v}$ and $\mathbf{u} \mathbf{v}^T$ related (you might try an example in \mathbb{R}^2).

True or False (and explain)?

1. If A and B are row equivalent, then they have the same row reduced echelon form.
2. If vectors \mathbf{v}_1 and \mathbf{v}_2 are linearly dependent, it is still possible that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent for some \mathbf{v}_3 .
3. A mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if each vector in \mathbb{R}^n maps onto a unique vector in \mathbb{R}^m .
4. $A^T + B^T = (A + B)^T$
5. $(AB)C = (AC)B$
6. If A is 5×5 , and the columns of A do not span \mathbb{R}^5 , it is possible that A is invertible.
7. A linear transformation preserves the operations of vector addition and scalar multiplication.
8. If $A\mathbf{x} = \mathbf{b}$ has more than 1 solution, so does $A\mathbf{x} = \mathbf{0}$.
9. In some cases, it is possible for four vectors to span \mathbb{R}^5 .
10. If A, B are row equivalent $m \times n$ matrices, and if the columns of A span \mathbb{R}^m , then so do the columns of B .
11. The span $\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.