

Example Questions, Exam 3, Math 240

1. Let A and its RREF be given as:

$$A = \begin{bmatrix} -1 & -5 & 3 & 9 \\ -48 & -40 & 24 & 92 \\ 94 & 70 & -42 & -166 \\ -48 & -40 & 24 & 92 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 10 & -3 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We also note two facts: $\lambda = 4$ is an eigenvalue of A , and $\mathbf{u} = [1, 0, 2, 0]^T$ is an eigenvector of A .

- (a) Find a basis for the eigenspace E_4 :
- (b) What is the eigenvalue for the eigenvector \mathbf{u} ?
- (c) You might have noticed that the second and fourth rows are the same. Does that imply we have a certain eigenvalue? Find a basis for its eigenspace. To save you some time, we have included the RREF of A .
- (d) What is the characteristic polynomial of A ?
- (e) Show that A is diagonalizable by finding an appropriate P and D .

2. Short Answer:

- (a) Show that if A^2 is the zero matrix, the only eigenvalue of A is zero.
- (b) Write the complex number in $a + ib$ form: $\frac{1-3i}{2+i}$.
- (c) Write the complex number in polar form, $re^{i\theta}$: $-1 - 3i$
- (d) Normalize the vector $[1, -2, 1, 1]^T$.
- (e) Suppose A is 3×3 , and \mathbf{u} is an eigenvector of A corresponding to an eigenvalue of 7.
Is \mathbf{u} an eigenvector of $2I - A$? If so, find the corresponding eigenvalue. If not, explain why not.
- (f) True or False? A matrix with orthonormal columns is an orthogonal matrix.

3. Show the following: If U, V are orthogonal matrices, then so is UV .

4. Let $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$, diagonalize the matrix A .

5. If each row of the matrix A sums to the same number r , and A is $n \times n$, then what must one eigenvalue of A be, and what eigenvector? (Hint: Is there a vector \mathbf{v} so that $A\mathbf{v}$ is a vector of row sums?)

6. Let U be $m \times n$ with orthonormal columns. Show that $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

Let U be $m \times n$ with orthonormal columns. Show that the angle between \mathbf{x} and \mathbf{y} is the same as the angle between $U\mathbf{x}$ and $U\mathbf{y}$.

7. Show that the eigenvalues of A and A^T are the same.

8. Show that if $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are eigenvectors corresponding to distinct eigenvalues, then the vectors are linearly independent.

9. Find the eigenvalues and bases for the eigenspaces if $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

10. Compute an appropriate factorization for the matrix $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

11. Let matrix A be $m \times n$. Show that the row space is orthogonal to the null space.

12. If $\mathbf{u} = [3, 2, -5, 0]$ and $\mathbf{v} = [1, 1, -1, 2]$, then compute:

- (a) The distance between \mathbf{u} and \mathbf{v} .
 (b) The angle between \mathbf{u} and \mathbf{v} (leave in exact form).
 (c) The orthogonal projection of \mathbf{u} onto \mathbf{v} .
 (d) Having an orthogonal projection means what two vectors are orthogonal? Show that this is indeed the case.
13. Prove the Pythagorean Theorem for two vectors \mathbf{x} and \mathbf{y} . If \mathbf{x} and \mathbf{y} are orthogonal vectors in \mathbb{R}^n , then
- $$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$
14. If A is similar to B , show that they have the same eigenvalues. Do they also have the same eigenvectors?
15. Prove that if the set $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ form a basis for subspace W , and \mathbf{x} is orthogonal to each \mathbf{v}_i , for $i = 1$ to k , then \mathbf{x} is orthogonal to W . (Hint: Start with a generic vector $\mathbf{w} \in W$, and show that $\mathbf{x} \cdot \mathbf{w} = 0$.)
16. Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.
17. (More) True or False? If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.
- (a) If \mathbf{v}_1 and \mathbf{v}_2 are linearly independent eigenvectors, then they correspond to distinct eigenvalues.
 (b) If A is invertible, then A is diagonalizable.
 (c) The orthogonal projection of \mathbf{y} onto a vector \mathbf{v} is the same as the orthogonal projection of \mathbf{y} onto $c\mathbf{v}$ whenever $c \neq 0$.
 (d) If A is an orthogonal matrix, then A^T is an orthogonal matrix.
 (e) If A, B have the same eigenvalues, then they are similar.
18. Show (by using the projection formula for one vector onto another) that if U is a matrix with orthonormal columns, then $UU^T\mathbf{x}$ is the projection of \mathbf{x} onto the columns of U .
 (NOTE: This formula also tells us that the coordinates of \mathbf{x} with respect to the columns of U is given by $U^T\mathbf{x}$).
 Similarly, show that, if U has orthonormal columns, then $U^TU = I$.
19. When we originally partitioned \mathbb{R}^n into one part as the row space and the other part as the null space for a matrix, we said that the only thing in both spaces was the zero vector. How do we know that there is not a vector in \mathbb{R}^n that is in neither the row space nor the null space? (Think about 6.3- the orthogonal decomposition theorem, for example).
20. Use Gram-Schmidt to give us an orthogonal set of vectors to replace the columns of the matrix A below.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

21. Let $A = QR$.

- (a) If A is 4×3 , what dimensions could Q and R be, if we use Gram-Schmidt to compute Q ?
 (b) Here are matrices A, Q and R . See if you can tell which is which.

$$\begin{bmatrix} 1 & -3 & 1/2 \\ 0 & -1 & -5/2 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \quad \begin{bmatrix} -1 & -3 & 1 \\ 3 & -1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

- (c) If $A = QR$, as given in the previous part, then write \mathbf{a}_3 as a linear combination of $\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3$.
 (d) If $A\mathbf{x} = \mathbf{b}$, and $A = QR$, then write the solution \mathbf{x} in terms of Q, R .