Extra Examples, Section 4.3

31. Let $T : V \to W$ be a linear transformation from vector space V into vector space W.

    Show that, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent in V, then $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent in W.

    SOLUTION: Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent, there exists number $c_1, c_2, \ldots, c_k$, not all zero, so that

    $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

    Taking T of both sides and using the linearity of T, we then have:

    $$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p) = \mathbf{0}$$

    And since $c_1, c_2, \cdots, c_p$ are not all zero, then the set of vectors $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent in W.

32. We want to show that if T is 1-1, then the set of images $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent, then the original set of vectors, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

    SOLUTION: What is wrong with the following logic (put down a correct argument for your solution)?

    We are told that $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_p)\}$ is linearly dependent in W. Therefore,

    $$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p) = \mathbf{0}$$

    And by the linearity of T,

    $$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = \mathbf{0}$$

    Therefore,

    $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

    where $c_1, c_2, \cdots, c_k$ are not all zero. Therefore, we can conclude that the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

    (Hint: I did not use the fact that T was 1-1...)

33. (This problem is not 33, but is very similar to 33) Is the set of vectors

    $$p_1(t) = 2t - t^2, \qquad p_2(t) = 2 + 2t \qquad p_3(t) = 2 + 8t - 3t^2$$

    linearly independent vectors in $\mathbb{P}_2$?

    SOLUTION: Use the definition of linear independence. That means, we look for constants C_1, C_2, C_3, not all zero, so that

    $$C_1(2t - t^2) + C_2(2 + 2t) + C_3(2 + 8t - 3t^2) = 0 \quad \text{for all } t$$

I'm going to multiply this out and re-order the terms so that it is of the form:

$$A_1 t^2 + A_2 t + A_3 = 0, \qquad \text{for all } t$$

This will mean that $A_1 = 0$, $A_2 = 0$, and $A_3 = 0$. You probably did something like this before when you solved for the coefficients in a partial fraction problem, or when you looked at power series.

In this case, we have:

$$(-C_1 - 3C_3)t^2 + (2C_1 + 2C_2 + 8C_3)t + (2C_2 + 2C_3) = 0 \qquad \text{for all } t$$

Equating the coefficients to zero, we get three equations in three unknowns:

$$\begin{array}{rrrl} -C_1 & & -C_3 & = 0 \\ 2C_1 & +2C_2 & +8C_3 & = 0 \\ & 2C_2 & +2C_3 & = 0 \end{array} \qquad \Rightarrow \qquad \begin{bmatrix} -1 & 0 & -3 \\ 2 & 2 & 8 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

There are an infinite number of (non-zero) solutions to our equation, so the three vectors (polynomials) are linearly dependent. In fact, you might check that

$$p_3(t) = 3p_1(t) + p_2(t)$$

Did you notice a relationship between the coefficients of each polynomial and the columns of our matrix?