Linear Algebra- Final Exam Review Questions

For the final exam, you may not use a calculator. The length of the exam is about an one and a half exams, and will be approximately weighted: 40% new material, then about 20% taken from the other three portions of the course (to add to 60%).

1. Show that $\mathrm{Row}(A) \perp \mathrm{Null}(A)$.

2. Let A be invertible. Show that, if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors, so are $A\mathbf{v}_1, A\mathbf{v}_2, A\mathbf{v}_3$. NOTE: It should be clear from your answer that you know the definition.

3. Find the line of best fit for the data:

x	0	1	2	3
y	1	1	2	2

4. Let $A = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix}$. (a) Is A orthogonally diagonalizable? If so, orthogonally diagonalize it! (b) Find the SVD of A.

5. Let V be the vector space spanned by the functions on the interval $[-1, 1]$.
$$\{1, t, t^2\}$$
Use Gram-Schmidt to find an orthonormal basis, if we define the inner product:
$$\langle f(t), g(t) \rangle = \int_{-1}^{1} 2f(t)g(t)\, dt$$

6. Suppose that vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent vectors in $\mathbb{R}^n$. Determine if the set $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}-2\mathbf{v}+\mathbf{w}\}$ are also linearly independent.

7. Let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be orthonormal. If
$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$
then show that $\|\mathbf{x}\|^2 = |c_1|^2 + \cdots + |c_p|^2$. (Hint: Write the norm squared as the dot product).

8. Short answer:
   (a) If $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}, \mathbf{v}$ are orthogonal.
   (b) Let H be the subset of vectors in $\mathbb{R}^3$ consisting of those vectors whose first element is the sum of the second and third elements. Is H a subspace?
   (c) Explain why the image of a linear transformation $T : V \to W$ is a subspace of W
   (d) Is the following matrix diagonalizable? Explain. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 8 \\ 0 & 0 & 13 \end{bmatrix}$

(e) If the column space of an 8×4 matrix A is 3 dimensional, give the dimensions of the other three fundamental subspaces. Given these numbers, is it possible that the mapping $\mathbf{x} \to A\mathbf{x}$ is one to one? onto?

(f) i. Suppose matrix Q has orthonormal columns. Must $Q^T Q = I$?

ii. True or False: If Q is $m \times n$ with $m > n$, then $QQ^T = I$.

iii. Suppose Q is an orthogonal matrix. Prove that $\det(Q) = \pm 1$.

9. Find a basis for the null space, row space and column space of A, if $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 5 & 5 \\ 0 & 0 & 3 & 3 \end{bmatrix}$

10. Find an orthonormal basis for $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ using Gram-Schmidt (you might wait until the very end to normalize all vectors at once):

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

11. Let $\mathbb{P}_n$ be the vector space of polynomials of degree n or less. Let W_1 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(0)\mathbf{p}(1) = 0$. Let W_2 be the subset of $\mathbb{P}_n$ consisting of $\mathbf{p}(t)$ so that $\mathbf{p}(2) = 0$. Which if the two is a subspace of $\mathbb{P}_n$?

12. For each of the following matrices, find the characteristic equation, the eigenvalues and a basis for each eigenspace:

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

13. Define $T : P_2 \to \mathbb{R}^3$ by: $T(p) = \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}$

(a) Find the image under T of $p(t) = 5 + 3t$.

(b) Show that T is a linear transformation.

(c) Find the kernel of T. Does your answer imply that T is $1-1$? Onto? (Review the meaning of these words: kernel, one-to-one, onto)

14. Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ so that $\|\mathbf{v}\| = 1$, and let $Q = I - 2\mathbf{v}\mathbf{v}^T$. Show (by direct computation) that $Q^2 = I$.

15. Let A be $m \times n$ and suppose there is a matrix C so that $AC = I_m$. Show that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$. Hint: Consider $AC\mathbf{b}$.

16. If B has linearly dependent columns, show that AB has linearly dependent columns. Hint: Consider the null space.

17. If λ is an eigenvalue of A, then show that it is an eigenvalue of A^T.

18. Let $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, Let S be the parallelogram with vertices at $\mathbf{0}, \boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{u} + v$. Compute the area of S.

19. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, $B = \begin{bmatrix} a+2g & b+2h & c+2i \\ d+3g & e+3h & f+3i \\ g & h & i \end{bmatrix}$, and $C = \begin{bmatrix} g & h & i \\ 2d & 2e & 2f \\ a & b & c \end{bmatrix}$.
If $\det(A) = 5$, find $\det(B), \det(C), \det(BC)$.

20. Let $1, t$ be two vectors in $C[-1,1]$. Find the length between the two vectors and the cosine of the angle between them using the standard inner product (the integral). Find the orthogonal projection of t^2 onto the set spanned by $\{1, t\}$.

21. Define an *isomorphism*. Why is an isomorphism important?

22. Let
$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 7 \end{bmatrix} \right\}$$
Find at least two $\mathcal{B}-$coordinate vectors for $\mathbf{x} = [1,1]^T$.

23. Let U, V be orthogonal matrices. Show that UV is an orthogonal matrix.

24. In terms of the four fundamental subspaces for a matrix A, what does it mean to say that:

    - $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
    - $A\mathbf{x} = \mathbf{b}$ has no solution.
    - In the previous case, what is the "least squares" solution? What quantity is being minimized?
    - $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions.

25. Let T be a one-to-one linear transformation for a vector space V into $\mathbb{R}^n$. Show that for $\mathbf{u}, \mathbf{v}$ in V, the formula:
$$\langle u, v \rangle = T(\mathbf{u}) \cdot T(\mathbf{v})$$
defines an inner product on V.

26. Describe all least squares solutions to $\begin{aligned} x + y &= 2 \\ x + y &= 4 \end{aligned}$

27. Let $\mathbf{u} = [5, -6, 7]^T$. Let W be the set of all vectors orthogonal to $\mathbf{u}$. (i) Geometrically, what is W? (ii) Find the projection of $\mathbf{x} = [1,2,3]^T$ onto W. (iii) Find the distance from the vector $\mathbf{x} = [1,2,3]^T$ to the subspace W.

28. Can the SVD be used to determine if a matrix A is invertible? How?