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- Question: What is the long term behavior of the orbit?

$$\lim_{t \to \infty} x_t = \lim_{t \to \infty} f^{(t)}(x_0)$$

Example 1

Let $x_0 = \frac{1}{2}$, and $f(x) = x^2$. Then:

i	0	1	2	3	$\cdots$
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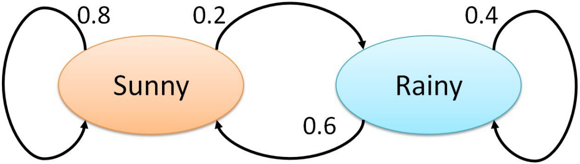
Example 2: Markov Chains

Markov chains can be used to simulate transitions between states using probabilities.



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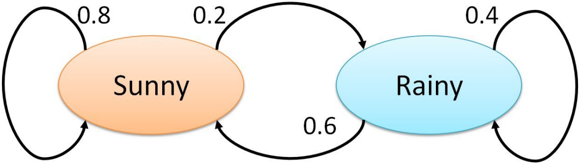

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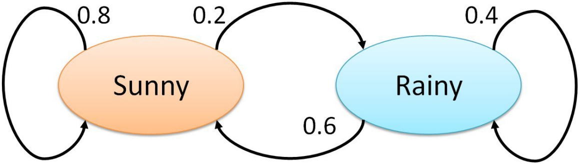

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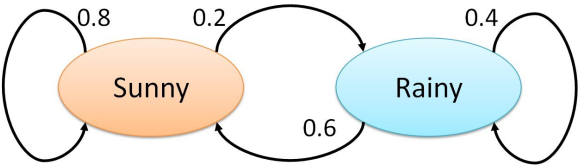

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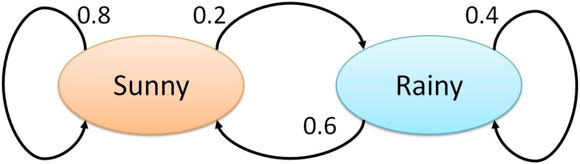

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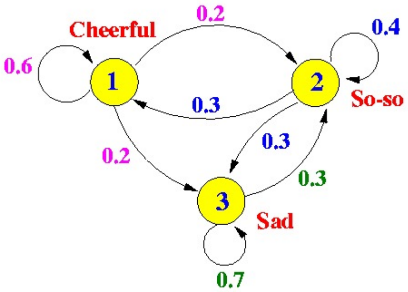
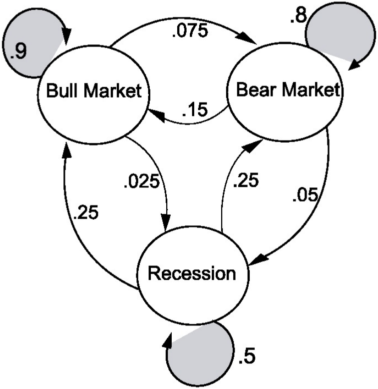


You can track your moods...

- C C - - C C C C C - S S - - C C C C C C



You might model the economy...

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$$M = \begin{array}{r|ccc} & & \text{From} & \\ & 1 & 2 & 3 \\ \hline 1 & 0.7 & 0.3 & 0.1 \\ \text{To: } 2 & 0.2 & 0.5 & 0.0 \\ 3 & 0.1 & 0.2 & 0.9 \end{array}$$

  The 0.3 entry means there is a probability of 30% of going to state 1 from state 2.
- A stochastic matrix is *regular* if there is a k so that P^k has all non-negative values. (The example is regular, if you check P^2).

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The dynamics: $\mathbf{x}_k = M^k \mathbf{x}_0$.

If 60% of the population is in the city, 40% in the suburbs, we can compute the percentages for next year:

$$\mathbf{x}_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = M\mathbf{x}_0 = \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix}$$

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$$\mathbf{x} = M\mathbf{x} \qquad M\mathbf{x} - \mathbf{x} = \vec{0} \qquad (M - I)\mathbf{x} = \vec{0}$$

For example,

$$(M - I)\mathbf{x} = \vec{0} \quad \Rightarrow \quad \begin{bmatrix} 0.95 - 1 & 0.03 \\ 0.05 & 0.97 - 1 \end{bmatrix} \mathbf{x} = \vec{0}$$

$$\begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \mathbf{x} = \vec{0}$$

Therefore, if $\mathbf{x} = [c, s]^T$, then:

$$-5c + 3s = 0 \quad \Rightarrow \quad \begin{aligned} c &= 3/5s \\ s &= s \end{aligned}$$

$$= s \begin{bmatrix} 3/5 \\ 1 \end{bmatrix} = s_1 \begin{bmatrix} 3 \\ 5 \end{bmatrix} = s_2 \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

If 3/8 of the population is in the city, 5/8 in the suburbs, then the populations will remain unchanged in time.

HOMEWORK: Exercises 1, 2, 3, 9, 11, 12