Review Solutions for Exam 1

Definitions and Basic Theorems

1. Finish the definition:

   (a) A **linear combination** of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is: any vector of the form

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

   (b) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ are said to be **linearly independent** if: the only solution to

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = 0$$

   is the trivial solution, $c_1 = c_2 = \cdots = c_n = 0$.

   (c) The **span** of a set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is: the set of all linear combinations of the vectors.

   (d) A system of equations is **inconsistent** if: there is no solution that satisfies all of the equations.

   (e) A system of equations is **homogeneous** if: the equations are set to zero.

   (f) Two matrices are **row equivalent** if: you can change one into the other by a sequence of row operations.

   (g) A transformation $T : X \to Y$ is said to be **linear** if: $T(\mathbf{x}_1 + \mathbf{x}_2) = T(\mathbf{x}_1) + T(\mathbf{x}_2)$, for all $\mathbf{x}_1, \mathbf{x}_2$ in X, and $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $\mathbf{x} \in X$ and all scalars c.

   (h) A transformation $T : X \to Y$ is said to be **one to one** if the preimage of every $\mathbf{y} \in Y$ is either empty or a single value of $\mathbf{x} \in X$.
   Alternative: If $T(x_1) = T(x_2)$, then $x_1 = x_2$.
   Altnerative: If $x_1 \neq x_2$, then $T(x_1) \neq T(x_2)$.

   (i) The definition of $A\mathbf{x}$: This is a linear combination of the columns of A using the weights in $\mathbf{x}$. If A is $m \times n$, then:

   $$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

2. What information about $T : \mathbb{R}^n \to \mathbb{R}^m$ do we need to know in order to compute the standard matrix for the transformation?

   SOLUTION: The best information to know is where T sends the standard basis vectors. That is, $T(\mathrm{e}_1), \ldots, T(\mathrm{e}_n)$. That's because these are the columns of the matrix. It is possible to use any linearly independent set of n vectors, but we would have some work to do to find the matrix.

3. Fill in the blanks for the Existence and Uniqueness Theorem. Your answers should refer to pivot columns:

- A linear system is consistent if and only if: the rightmost column of the augmented matrix is not a pivot column.
- Furthermore, if the system is consistent, the solution is unique if: every column of the coefficient matrix is a pivot column.

4. Give three statements that are logically equivalent to saying that A has a pivot in every row. I'll give some hints so you can fill in the blanks:

   - For each $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for every $\mathbf{b}$.
   - The columns of A span $\mathbb{R}^m$
   - Each $\mathbf{b}$ is a linear combination of the columns of A
   - The mapping $\mathbf{x} \to A\mathbf{x}$ will be ONTO.

5. Similar to the last problem, give two statements that are logically equivalent to saying that: A has a pivot in every column.

   SOLUTION: Here are some options:

   (a) The mapping $\mathbf{x} \to A\mathbf{x}$ is 1-1.
   (b) For every $\mathbf{b} \in \mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has AT MOST 1 solution.
   (c) The homogeneous equation has only the trivial solution.
   (d) The columns of A are linearly independent.

6. Suppose A is $m \times n$ with $m > n$. Is it possible that the mapping $\mathbf{x} \to A\mathbf{x}$ is 1-1? Onto? (Explain).

   SOLUTION: If $m > n$ and we have the maximum number of pivots (n of them), then every column is a pivot column. This means that the equation

   $$A\mathbf{x} = \mathbf{b}$$

   has *at most* one solution for every $\mathbf{b}$. That means the mapping $\mathbf{x} \to A\mathbf{x}$ could be $1 - 1$ (it is possible, but not necessary).

   On the other hand, if we have the maximum number of pivots, we would still not have enough for all the rows of A, thus we must have at least one row of A that is not a pivot row. That means that we can find $\mathbf{b}$ so that

   $$A\mathbf{x} = \mathbf{b}$$

   has no solution, and so the mapping can never be "onto".

7. Same question, but let A be $m \times n$ with $m < n$.

   SOLUTION: With the matrix "wide" instead of "tall", we might be able to get a pivot in every row, but we will never get a pivot in every column. Thus, while it is possible that the equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ (meaning it is possible that the mapping $\mathbf{x} \to A\mathbf{x}$ is "onto"), the solution will never be unique (for any $\mathbf{b}$). Therefore, the mapping $A\mathbf{x} = \mathbf{b}$ cannot be $1 - 1$.

Computational Questions

1. Find the row reduced echelon form of the matrix A given below. Be sure to show all your work:

   SOLUTION:

$$A = \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. If the matrix given above was actually an augmented matrix, use your row reduced echelon form to give the solution to the system.

   SOLUTION: If this had been an augmented matrix, then the RREF would be:

$$\left[\begin{array}{cccc|c} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{lllll} x_1 & = 5 & +7x_2 & -6x_4 \\ x_2 & = & x_2 & \\ x_3 & = -3 & & +2x_4 \\ x_4 & = & & x_4 \end{array} \rightarrow \mathbf{x} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

3. Do the three lines $x_1 - 4x_2 = 1, 2x_1 - x_2 = -3$ and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

   SOLUTION: If there is a common point of intersection, it would be found by solving the given system of equations (and there would be a unique solution):

$$\left[\begin{array}{cc|c} 1 & -4 & 1 \\ 2 & -1 & -3 \\ -1 & -3 & 4 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -4 & 1 \\ 0 & 7 & -5 \\ 0 & 0 & 0 \end{array}\right]$$

   Every column is a pivot column, and the system is consistent, so the solution is unique, and the three lines do indeed have a point of intersection.

4. Let $\mathbf{a}_3 = 2\mathbf{a}_1 - 3\mathbf{a}_2$. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$. If A is 3×3 with 2 pivots, write the solution to $A\mathbf{x} = \mathbf{0}$ in parametric form.

   SOLUTION: If we have two pivots, then there is one free variable. Therefore, we know the solution set is a line through the origin. Also, since

$$2\mathbf{a}_1 - 3\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$$

   then the solution set must be: $s[2, -3, -1]^T$, where s is a free value (parameter).

5. Write the equation of a plane that spans $[1, 2, 3, 4]^T$, $[1, -1, -1, 1]^T$ and has been translated by $[3, 0, 0, 1]^T$

   SOLUTION:

$$\begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

6. Find the general solution (in parametric vector form) to the system:

$$\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= -1 \\ -2x_1 - 6x_2 - x_3 &= 5 \\ x_1 + 3x_2 + 2x_3 + 3x_4 &= 2 \end{aligned}$$

SOLUTION: Row reduction gets you the matrix to the left; solving gives you the parametric equation on the right:

$$\left[\begin{array}{rrrr|r} 1 & 3 & 1 & 1 & -1 \\ -2 & -6 & -1 & 0 & 5 \\ 1 & 3 & 2 & 3 & 2 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 3 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boldsymbol{x} = \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

7. Suppose the solution set of a certain system of linear equations is given by $x_1 = 5+4x_4$, $x_2 = -2 + 7x_4$ with $x_3 = 2 + x_4$ and x_4 is a free variable.

   (a) Use vectors to describe the solution set as a (parametric) line in $\mathbb{R}^4$.
   SOLUTION: $\mathbf{x} = [5, -2, 2, 0]^T + x_4[4, 7, 1, 1]^T$

   (b) Was the original system homogeneous? If not, give the solution to the homogeneous system of equations, if you have enough information.

   SOLUTION: No, the system was not homogeneous since 0 is not a solution. However, we know all solutions are of the form $\mathbf{x}_p + \mathbf{x}_h$, where the parameters are in $\mathbf{x}_h$. Therefore, we can think of $[5, -2, 2, 0]^T$ as the particular part of the solution to $A\mathbf{x} = \mathbf{b}$, and $x_4[4, 7, 1, 1]^T$ is the homogeneous part of the solution.

8. Show that the mapping T is not linear: $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 - 2x_2 \\ 3x_2 + 1 \end{bmatrix}$

   Quick method: $T(\vec{0}) = [0, 1]^T \neq [0, 0]^T$, therefore the mapping is not linear.

9. Determine if the mapping is linear: $T(x_1, x_2) = x_1^2 + 3x_2$.

   SOLUTION: We check if $T(x_1 + y_1, x_2 + y_2) = T(x_1, x_2) + T(y_1, y_2)$:

$$T(x_1 + y_1, x_2 + y_2) = (x_1 + y_1)^2 + 3(x_2 + y_2)$$

   And

$$T(x_1, x_2) + T(y_1, y_2) = (x_1^2 + y_1^2) + 3(x_2 + y_2)$$

   These two quantities are not the same- The first quantity has $2x_1y_1$ term that the second does not. Therefore, T is not linear.

   Alternative: Check that $T(cx_1, cx_2) = cT(x_1, x_2)$

$$T(cx_1, cx_2) = (cx_1)^2 + 3cx_2 = c^2x_1^2 + c3x_1 = c(cx_1^2 + 3x_2) \neq cT(x_1, x_2)$$

10. Given the matrix A below, explain whether or not the system $A\mathbf{x} = \mathbf{b}$ has a solution in terms of h. If there are restrictions on $\mathbf{b}$, give them.

$$A = \begin{bmatrix} 1 & -3 \\ 2 & -h \end{bmatrix}$$

SOLUTION: Row reduce:

$$\left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 2 & -h & b_2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & b_1 \\ 0 & 6-h & b_2 - 2b_1 \end{array}\right]$$

Therefore, if $h \neq 6$, we have a unique solution for all vectors $\mathbf{b}$.

If $h = 6$, the system is consistent only when $b_2 = 2b_1$, and in that case we have an infinite number of solutions to the system. Otherwise, if $b_2 \neq 2b_1$, the system is inconsistent.

11. Let A be a 3×4 matrix, let $\boldsymbol{y}_1, \boldsymbol{y}_2$ be vectors, and let $\boldsymbol{w} = \boldsymbol{y}_1 + \boldsymbol{y}_2$. Suppose that $\boldsymbol{y}_1 = A\boldsymbol{x}_1$ and $\boldsymbol{y}_2 = A\boldsymbol{x}_2$ for some vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$.

   (a) What size must $\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_1, \boldsymbol{x}_2$ be?
   SOLUTION: The vectors $\mathbf{x}_1, \mathbf{x}_2$ are in $\mathbb{R}^4$ and $\mathbf{y}_1, \mathbf{y}_2$ are in $\mathbb{R}^3$

   (b) Does $A\boldsymbol{x} = \boldsymbol{w}$ have a solution? Why or why not?
   SOLUTION: Yes-

$$\mathbf{w} = \mathbf{y}_1 + \mathbf{y}_2 = A\mathbf{x}_1 + A\mathbf{x}_2 = A(\mathbf{x}_1 + \mathbf{x}_2)$$

   so the solution to $A\mathbf{x} = \mathbf{w}$ is $\mathbf{x} + \mathbf{x}_1 + \mathbf{x}_2$.

12. Suppose that:

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \quad \text{and } T\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Find a matrix A so that $T(\boldsymbol{x}) = A\boldsymbol{x}$.

SOLUTION: In this case, we don't know $T(\mathbf{e}_1, T(\mathbf{e}_2)$, so we'll try something different. Let the matrix A be given below, so that we need to find a, b, c, d:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

These lead us to the system of equations:

$$\begin{aligned} a + 2b &= -3 \\ -2a + b &= 1 \end{aligned} \qquad \begin{aligned} c + 2d &= 0 \\ -2c + d &= 2 \end{aligned} \quad \Rightarrow \quad A = \begin{bmatrix} -1 & -1 \\ -4/5 & 2/5 \end{bmatrix}$$

13. Determine the matrix for the linear transformation T given below: $T(x_1, x_2, x_3, x_4) = 3x_1 - 4x_2 + 8x_4$

    SOLUTION: You could think about $T(\mathbf{e}_i)$, or just reason it out. Note that $T : \mathbb{R}^4 \to \mathbb{R}$, so A is 1×4, and

$$A = [3, -4, 0, 8]$$

14. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$, where $T(\boldsymbol{e}_1) = (1, 4)$, $T(\boldsymbol{e}_2) = (-2, 9)$, and $T(\boldsymbol{e}_3) = (3, -8)$. Find a matrix A so that $T(\mathbf{x}) = A\mathbf{x}$.

    SOLUTION: First, note the size of A should be 2×3, and we already have the action of T on the standard basis vectors. The matrix is $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 9 & -8 \end{bmatrix}$

15. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ so that $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \\ -1 & -3 \end{bmatrix}$$

    Is T 1-1? Explain. Is T onto? Explain.

    SOLUTION: Examining the matrix, we see that there are two pivots. Therefore, if the system is consistent (and the last row of the rref form of A will be all zeros), there is exactly one solution.

    This corresponds to the mapping being $1-1$ but not onto- That is, the solution to $A\mathbf{x} = \mathbf{b}$, if it exists, is unique (but it may not exist).

16. Suppose we want to determine a quadratic function $f(x) = a_0 + a_1 x + a_2 x^2$ that interpolates the data $(1, 1)$, $(-1, 2)$, and $(2, 3)$. Write down the system of equations (and the corresponding matrix equation) we need to solve (do not actually solve the system) to find the polynomial.

    SOLUTION: Put each point in turn into the given polynomial. That gives the system of equations.

$$\begin{array}{llll} a_0 & +a_1(1) & +a_2(1)^2 & = 1 \\ a_0 & +a_1(-1) & +a_2(-1)^2 & = 2 \\ a_0 & +a_1(2) & +a_2(2)^2 & = 3 \end{array} \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \\ 1 & 2 & 4 & 3 \end{array}\right]$$

Discussion Questions

1. Suppose A is 3×3 an $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does not have a solution. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{z}$ has a unique solution?

    SOLUTION: The setup is telling us that the RREF of the coefficient matrix A has a row of zeros. Therefore, there are at most two pivots, and we have three columns. Therefore, at least one column of A is not a pivot column, so if the equation is consistent, we must have an infinite number of solutions (we have at least one free variable).

2. Let A be $n \times n$. If the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, do the columns of A span $\mathbb{R}^n$? Why or why not? Is your answer different if A is $n \times m$?

   SOLUTION: Yes. If the equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution, there is a pivot in every column. Since there are n columns, we have n pivots, so there is a pivot in every row, and therefore, $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$, and therefore the columns of A span $\mathbb{R}^n$.

   The answer is false if A is not square. Such a matrix would necessarily be tall, and the columns would not span the appropriate space.

3. Let T be a linear transformation. Show that if $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ are linearly dependent vectors, then $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)\}$ are linearly dependent vectors.

   SOLUTION: If the set of vectors is linearly dependent, there is a non-trivial solution to

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3 = \mathbf{0}$$

   Apply T to both sides and use the linearity of T:

$$T(c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + c_3\boldsymbol{v}_3) = T(\mathbf{0}) \Rightarrow c_1T(\boldsymbol{v}_1) + c_2T(\boldsymbol{v}_2) + c_3T(\boldsymbol{v}_3) = \mathbf{0}$$

   This equation says that there is a non-trivial solution, so the vectors $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), T(\boldsymbol{v}_3)\}$ are linearly dependent vectors.

4. If H is 7×7 matrix and $H\boldsymbol{x} = \boldsymbol{v}$ is consistent for every $\boldsymbol{v}$ in $\mathbb{R}^7$, then is it possible for $H\boldsymbol{x} = \boldsymbol{v}$ to have *more* than one solution for some $\boldsymbol{v} \in \mathbb{R}^7$? Why or why not?

   SOLUTION: If the given equation has a solution for every $\mathbf{v}$, then there is a pivot in every row of H. Since there are the same number of rows as columns, H also must have a pivot in every column, and so there can not be more than one solution for any $\mathbf{v}$.

5. Suppose that the third column of B is the sum of the first two columns, which are not linear combinations of each other. If B is 4×3, give the matrix which should be the RREF of B.

   SOLUTION: The first two columns are basic variables, the third is not. Also, from the information given, we can find the values of the last column to be:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

6. Suppose that the full solution to $A\mathbf{x} = \boldsymbol{b}$ is given by

$$\mathbf{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

(a) Give the RREF of A, if A is 3×3.

SOLUTION: We note from the position of the 1's in the last two vectors, that c_1 is actually x_2 and c_2 is actually x_3. Therefore:

$$\begin{array}{lllll} x_1 & = 4 & +2x_2 & +5x_3 \\ x_2 & = & x_2 & \\ x_3 & = & & x_3 \end{array} \qquad \Rightarrow \qquad \left[\begin{array}{rrr|r} 1 & -2 & -5 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(b) If the following row operations were applied to A (in order):

$$3R_1 + R_2 \rightarrow R_2 \qquad -5R_1 + R_3 \rightarrow R_3$$

Find the matrix A and the vector $\mathbf{b}$.

SOLUTION: To "invert" the operations, take the last one first, and perform the following to the RREF:

$$5R_1 + R_3 \rightarrow R_3 \qquad \text{and} \qquad -3R_1 + R_2 \rightarrow R_2$$

Resulting in the final matrix (the original $[A|b]$:

$$\left[\begin{array}{rrr|r} 1 & -2 & -5 & 4 \\ -3 & 6 & 15 & -12 \\ -5 & -10 & -25 & 20 \end{array}\right]$$

True or False (and explain)?

1. If A and B are row equivalent, then they have the same row reduced echelon form.

   TRUE- If A and B are row equivalent, then some (invertible) sequence of elementary row operations will take you from matrix A to B. Similarly, there is an (invertible) sequence of elementary row operations that will take A to its unique RREF.

2. If vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, it is still possible that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to be linearly independent for some $\mathbf{v}_3$.

   FALSE. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, then there is a nontrivial solution to the vector equation:

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \vec{0}$$

   This has the same (nontrivial) solution as:

   $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + 0 \cdot \mathbf{v}_3 = \vec{0}$$

   which, by definition, means that the full set of three vectors is linearly dependent.

3. A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each vector in $\mathbb{R}^n$ maps onto a unique vector in $\mathbb{R}^m$.

   FALSE, this is a statement of what it means to be a function. To be 1-1, each $\mathbf{y} \in \mathbb{R}^m$ must have come from a unique $\mathbf{x} \in \mathbb{R}^n$ (or have no solution).

4. If A is 5×5, and the columns of A do not span $\mathbb{R}^5$, it is possible that A is invertible.

   FALSE: There are several ways of describing what's happening here- For example,

   If the columns do not span $\mathbb{R}^5$, the mapping is not onto. Because A is square, the mapping is also not 1-1, and so the mapping is not invertible.

5. A linear transformation preserves the operations of vector addition and scalar multiplication.

   TRUE: This is another way to state the definition of linearity.

6. If $A\mathbf{x} = \mathbf{b}$ has more than 1 solution, so does $A\mathbf{x} = \mathbf{0}$.

   TRUE: True. Theorem 6 in Section 1.5 essentially says that when $A\mathbf{x} = \mathbf{b}$ is consistent, the solution sets of the non-homogeneous equation and the homogeneous equation are translates of each other. In this case, the two equations have the same number of solutions.

7. In some cases, it is possible for four vectors to span $\mathbb{R}^5$.

   FALSE: Construct the matrix A from the four vectors, so that A is 5×4. Then to say the vectors span $\mathbb{R}^5$ means that

   $$A\mathbf{c} = \mathbf{b}$$

   has a solution for every $\mathbf{b}$. But we can have at most 4 pivots (the number of columns), so the last row in the rref of A must be a row of zeros, so the last column could be a pivot column for some choices of $\mathbf{b}$.

8. If A, B are row equivalent $m \times n$ matrices, and if the columns of A span $\mathbb{R}^m$, then so do the columns of B.

   True. If the columns of A span $\mathbb{R}^m$, then the reduced echelon form of A is a matrix U with a pivot in each row (For more info, see Theorem 4 in Section 1.4). Since B is row equivalent to A, B can be transformed by row operations first into A and then further transformed into U. Since U has a pivot in each row, so does B. Therefore, the columns of B span $\mathbb{R}^m$.

9. The span$\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.

   FALSE. If the two vectors are both zero, the span is a point. If only one vector is non-zero, or if the two vectors are scalar multiples of each other, then the span is a line. If the two vectors are non-zero and are not multiples of each other, then the span can be visualized as a plane.