Example Questions, Exam 3, Math 240

1. Let A and its RREF be given as:

$$A = \begin{bmatrix} -1 & -5 & 3 & 9 \\ -48 & -40 & 24 & 92 \\ 94 & 70 & -42 & -166 \\ -48 & -40 & 24 & 92 \end{bmatrix} \qquad \text{rref}(A) = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 10 & -3 & -17 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We also note two facts: $\lambda = 4$ is an eigenvalue of A, and $\mathbf{u} = [1, 0, 2, 0]^T$ is an eigenvector of A (not necessarily with $\lambda = 4$).

   (a) Find a basis for the eigenspace E_4:

   (b) What is the eigenvalue for the eigenvector $\mathbf{u}$?

   (c) You might have noticed that the second and fourth rows are the same. Does that imply we have a certain eigenvalue? Find a basis for its eigenspace. To save you some time, we have included the RREF of A.

   (d) What is the characteristic polynomial of A?

   (e) Show that A is diagonalizable by finding an appropriate P and D.

2. Short Answer:

   (a) Show that if A^2 is the zero matrix, the only eigenvalue of A is zero.

   (b) Write the complex number in $a + ib$ form: $\frac{1-3i}{2+i}$.

   (c) Write the complex number in polar form, $re^{i\theta}$: $-1 - 3i$

   (d) Normalize the vector $[1, -2, 1, 1]^T$.

   (e) Suppose A is 3×3, and $\mathbf{u}$ is an eigenvector of A corresponding to an eigenvalue of 7.
   Is $\mathbf{u}$ an eigenvector of $2I - A$? If so, find the corresponding eigenvalue. If not, explain why not.

   (f) True or False? A matrix with orthonormal columns is an orthogonal matrix.

3. Show the following: If U, V are orthogonal matrices, then so is UV.

4. Let $A = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$, diagonalize the matrix A.

5. If each row of the matrix A sums to the same number r, and A is $n \times n$, then what must one eigenvalue of A be, and what eigenvector? (Hint: Is there a vector $\mathbf{v}$ so that $A\mathbf{v}$ is a vector of row sums?)

6. Let U be $m \times n$ with orthonormal columns. Show that $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

   Let U be $m \times n$ with orthonormal columns. Show that the angle between $\mathbf{x}$ and $\mathbf{y}$ is the same as the angle between $U\mathbf{x}$ and $U\mathbf{y}$.

7. Show that the eigenvalues of A and A^T are the same.

8. Show that if $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ are eigenvectors corresponding to distinct eigenvalues, then the vectors are linearly independent.

9. Find the eigenvalues and bases for the eigenspaces if $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

10. Compute an appropriate factorization for the matrix $A = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix}$.

11. Let matrix A be $m \times n$. Show that the row space is orthogonal to the null space.

12. If $\mathbf{u} = [3, 2, -5, 0]$ and $\mathbf{v} = [1, 1, -1, 2]$, then compute:

(a) The distance between $\mathbf{u}$ and $\mathbf{v}$.

(b) The angle between $\mathbf{u}$ and $\mathbf{v}$ (leave in exact form).

(c) The orthogonal projection of $\mathbf{u}$ onto $\mathbf{v}$

(d) Having an orthogonal projection means what two vectors are orthogonal? Show that this is indeed the case.

13. Prove the Pythagorean Theorem for two vectors $\mathbf{x}$ and $\mathbf{y}$. If $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors in $\mathbb{R}^n$, then

$$\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

14. If A is similar to B, show that they have the same eigenvalues. Do they also have the same eigenvectors?

15. Prove that if the set $\{\mathbf{v}_1, \cdots \mathbf{v}_k\}$ form a basis for subspace W, and $\mathbf{x}$ is orthogonal to each $\mathbf{v}_i$, for $i = 1$ to k, then $\mathbf{x}$ is orthogonal to W. (Hint: Start with a generic vector $\mathbf{w} \in W$, and show that $\mathbf{x} \cdot \mathbf{w} = 0$.)

16. Prove that the eigenvalues of a triangular matrix are the entries on its main diagonal.

17. (More) True or False? If the statement is false and you can provide a counterexample to demonstrate this, then do so. If the statement is false and be can slightly modified so as to make it true then indicate how this may be done.

    (a) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

    (b) If A is invertible, then A is diagonalizable.

    (c) The orthogonal projection of $\mathbf{y}$ onto a vector $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

    (d) If A is an orthogonal matrix, then A^T is an orthogonal matrix.

    (e) If A, B have the same eigenvalues, then they are similar.

18. Show (by using the projection formula for one vector onto another) that if U is a matrix with orthonormal columns, then $UU^T\mathbf{x}$ is the projection of $\mathbf{x}$ onto the columns of U.

    (NOTE: This formula also tells us that the coordinates of $\mathbf{x}$ with respect to the columns of U is given by $U^T\mathbf{x}$).

    Similarly, show that, if U has orthonormal columns, then $U^TU = I$.

19. When we originally partitioned $\mathbb{R}^n$ into one part as the row space and the other part as the null space for a matrix, we said that the only thing in both spaces was the zero vector. How do we know that there is not a vector in $\mathbb{R}^n$ that is in neither the row space nor the null space? (Think about 6.3- the orthogonal decomposition theorem, for example).

20. Use Gram-Schmidt to give us an orthogonal set of vectors to replace the columns of the matrix A below.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

21. Let $A = QR$.

    (a) If A is 4×3, what dimensions could Q and R be, if we use Gram-Schmidt to compute Q?

    (b) Here are matrices A, Q and R. See if you can tell which is which.

$$\begin{bmatrix} 1 & -3 & 1/2 \\ 0 & -1 & -5/2 \\ 0 & 0 & -1 \end{bmatrix} \qquad \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix} \qquad \begin{bmatrix} -1 & -3 & 1 \\ 3 & -1 & 1 \\ 1 & -1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

(c) If $A = QR$, as given in the previous part, then write $\mathbf{a}_3$ as a linear combination of $\mathbf{q}_1, .\mathbf{q}_2, \mathbf{q}_3$.

(d) If $A\mathbf{x} = \mathbf{b}$, and $A = QR$, then write the solution $\mathbf{x}$ in terms of Q, R.

(e) Suppose $A = PDP^{-1}$ with a suitable 2×2 matrix P and $D = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

i. If $B = 3I - 2A + A^2$, show that B is diagonalizable by finding an appropriate factorization of B.

ii. From your previous answer, if λ is an eigenvalue of A, then what would an eigenvalue of $A^2 + bA + cI$ be?

Additional Questions, 4.9- Markov Chains

1. (A calculator may be used on this problem. On the exam, I will choose numerical values that will be easy to work with, since you won't have a calculator).

   On any given day, a student is either healthy or ill. Of the students that are healthy today, 95% will be healthy tomorrow. Of the students that are ill today, 55% will be ill tomorrow.

   (a) What is the stochastic matrix for this system if we model it as a Markov chain?

   (b) Suppose 20% of the students are ill on Monday. What is the fraction or percentage of the students that will likely be ill on Tuesday?

   (c) What happens to the percentages in the long run?

2. Show that every 2×2 stochastic matrix has at least one steady-state vector.

   Hint: Any such matrix can be written in the form $P = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix}$, where α, β are constants between 0 and 1.

   Also, how many steady state vectors are there if $\alpha = \beta = 0$?

3. Let S be the $1 \times n$ row of all ones. If P is an $n \times n$ square matrix with all non-negative entries, show that P is stochastic if $SP = S$.