Review Material, After the Third Exam

The material after the third exam includes: 6.5 (Least Squares), 6.6 (Applications to Linear Models), 6.7 (Inner Product Spaces), then we looked at 7.1 (Symmetric Matrices and the Spectral Theorem), and finally we looked at 7.4 (The SVD).

Important Definitions

Orthogonal matrix, Least squares solution of $A\mathbf{x} = \mathbf{b}$, normal equation, symmetric matrix, orthogonally diagonalizable. Given a general inner product, be able to define the norm, the length, the distance and the angle for vectors in a general vector space. Define the singular values of a matrix, and the pseudoinverse.

Skills (Partial List)

Compute the least squares solution using the normal equations. Set up a matrix equation for a given linear model and find the least squares solution.

Be able to prove the Triangle Inequality (Theorem 6.7.17)- That is, be able to expand the norm in terms of the inner product and simplify.

Use Gram-Schmidt and be able to compute projections in a general vector space (using a given inner product).

Orthogonally diagonalize a symmetric matrix. Be able to project a vector into a subspace spanned by a given set of orthonormal vectors (and be able to use matrix notation). Understand how the Spectral Decomposition works (decomposition into rank one matrices).

Be able to compute the SVD by hand for small matrices. Understand the specific relationship between the SVD for a matrix A and the four fundamental subspaces. Be able to discuss solutions to $A\mathbf{x} = \mathbf{b}$ in terms of the SVD of A.

Important Theorems

6.5.14 (Solving Normal Eqns), 7.1.2 (Symmetric is equivalent to orth. diag.), 7.1.3 (The Spectral Theorem), 7.4.10 (The SVD).

Review Questions

1. Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, given A and $\mathbf{b}$ below. Note that the columns of A are orthogonal, and use that fact.

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

2. Find the line that best fits the data: $(-1,-1),(0,2),(1,4),(2,5)$. Do this by first finding a matrix equation that you will then find the least squares solution to (by using the normal equations).

3. Suppose A is $m \times n$ with linearly independent columns and $\mathbf{b}$ is in $\mathbb{R}^m$. Use the normal equations to produce a formula for $\hat{\mathbf{b}}$, the projection of $\mathbf{b}$ onto the column space of A. (Hint: First find $\hat{\mathbf{x}}$ which does not require an orthogonal basis for $\mathrm{Col}(A)$.)

4. Show that if $\mathbf{x} \in \mathrm{Null}(A)$, then $\mathbf{x} \in \mathrm{Null}(A^TA)$.

   Show that if $A^TA\mathbf{x} = 0$, then $\|A\mathbf{x}\| = ?$.

   Use the above to show that, if $\mathbf{x} \in \mathrm{Null}(A^TA)$, then $\mathbf{x} \in \mathrm{Null}(A)$.

   Altogether, this problem is showing that the null spaces of A and A^TA are the same!

5. Using the last problem, what can we conclude about the rank of A versus the rank of A^TA?

6. Suppose I have a model equation: $y = \beta_0 + \beta_1 \sin(v) + \beta_2 \ln(w)$.

   Given the following data, set up the matrix equation from which we could determine a least squares solution for the β's:

v	w	y
-1	2	1
1	1	2
0	3	-1
3	2	0

   (Do NOT actually solve for the β's, just set up the matrix equation).

7. Given vectors $\mathbf{u}, \mathbf{v}$ in the vector space $\mathbb{R}^n$ with the usual dot product as inner product, show that the Pythagorean Theorem still holds. That is, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other, then:

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

8. Orthogonally diagonalize the symmetric matrix $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$.

9. True or False, and explain: For every non-zero vector $\mathbf{v} \in \mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

10. Show that, if A is symmetric, then any two eigenvectors from distinct eigenvalues are orthogonal. Hint: Start with $\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2$, and see if you can transform this into $\lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2$.

11. Suppose we have the matrix $A = [1, 1, 1]$.

    (a) What will the singular values of A be? (Try to compute them in the easiest possible way).

    (b) Find (by hand) the reduced SVD for the matrix A. See if you can do it without any computation.

    (c) Find a basis for the null space of A using the rest of the SVD that hasn't been computed yet (this one we'll need to compute).

12. Show that the eigenvalues of $A^T A$ are non-negative. Hint: Consider $\|A\mathbf{v}_i\|$.

13. Suppose the SVD was given as the following:

$$A = \begin{bmatrix} 0.65 & -0.75 & 0 \\ 0 & 0 & 1 \\ 0.75 & 0.65 & 0 \end{bmatrix} \begin{bmatrix} 15.91 & 0 & 0 \\ 0 & 3.26 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.52 & -0.62 & -0.57 \\ -0.27 & 0.76 & -0.57 \\ -0.80 & 0.14 & 0.57 \end{bmatrix}^T$$

    (a) What is the rank of A?

    (b) Write a basis for the column space and null space of A.

    (c) Write the matrix product for the pseudoinverse of A (you don't need to multiply it out).

14. Suppose A is square and invertible. Find the SVD of A^{-1}.

15. Show that if A is square, then $|\det(A)|$ is the product of the singular values of A.