Solutions to the Review 4 Exercises

1. Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, given A and $\mathbf{b}$ below. Note that the columns of A are orthogonal, and use that fact.

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \\ 1 & -2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

SOLUTION: Since the columns of A are orthogonal, we can compute the $\hat{\mathbf{b}}$ directly.

$$\hat{\mathbf{b}} = \frac{\mathbf{b}^T\mathbf{a}_1}{\mathbf{a}_1^T\mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{b}^T\mathbf{a}_2}{\mathbf{a}_2^T\mathbf{a}_2}\mathbf{a}_2 = \frac{7}{9}\mathbf{a}_1 + \frac{1}{9}\mathbf{a}_2 = A\hat{\mathbf{x}}$$

so we can read $\hat{\mathbf{x}}$ off: $[7/9, 1/9]^T$. (See page 414 for another example).

2. Find the line that best fits the data: $(-1,-1), (0,2), (1,4), (2,5)$. Do this by first finding a matrix equation that you will then find the least squares solution to (by using the normal equations).

SOLUTION: The model equation is $y = \beta_0 + \beta_1 x$, so the matrix equation is:

$$\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

Forming the normal equations, we have:

$$A^TA\mathbf{c} = A^T\mathbf{y} \quad \Rightarrow \quad \begin{bmatrix} 3 & 1 \\ 1 & 10 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \frac{1}{29} \begin{bmatrix} 10 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 18 \end{bmatrix} = \frac{1}{29} \begin{bmatrix} 62 \\ 46 \end{bmatrix}$$

3. Suppose A is $m \times n$ with linearly independent columns and $\mathbf{b}$ is in $\mathbb{R}^m$. Use the normal equations to produce a formula for $\hat{\mathbf{b}}$, the projection of $\mathbf{b}$ onto the column space of A. (Hint: First find $\hat{\mathbf{x}}$ which does not require an orthogonal basis for Col(A).)

SOLUTION: Given $A\mathbf{x} = \mathbf{b}$, we know that $\hat{\mathbf{x}}$ solves the least squares problem:

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}$$

And that $\hat{\mathbf{b}} = A\hat{\mathbf{x}}$. Therefore, we get $\hat{\mathbf{b}}$ by multiplying both sides of our previous equation by A:

$$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = A(A^TA)^{-1}A^T\mathbf{b}$$

Therefore, the matrix that sends $\mathbf{b}$ to $\hat{\mathbf{b}}$ is $A(A^TA)^{-1}A^T$.

Side remark: If A was an invertible matrix, then this entire expression simplifies to I.

4. Show that if $\mathbf{x} \in \text{Null}(A)$, then $\mathbf{x} \in \text{Null}(A^T A)$.

   SOLUTION: If $\mathbf{x} \in \text{Null}(A)$, then $A\mathbf{x} = \mathbf{0}$. Multiplying both sides by A^T, we see that $A^T A\mathbf{x} = \mathbf{0}$, so that $\mathbf{x} \in \text{Null}(A^T A)$.

   Show that if $A^T A\mathbf{x} = 0$, then $\|A\mathbf{x}\| =?$.

   SOLUTION: Looking at the expression to the left, $\|A\mathbf{x}\|^2 = (A\mathbf{x} \cdot (A\mathbf{x}) = \mathbf{x}^T A^T A\mathbf{x}$. Now, if

$$A^T A\mathbf{x} = \mathbf{0}$$

   then

$$\mathbf{x}^T A^T A\mathbf{x} = 0 \quad \Rightarrow \quad \|A\mathbf{x}\|^2 = 0$$

   Use the above to show that, if $\mathbf{x} \in \text{Null}(A^T A)$, then $\mathbf{x} \in \text{Null}(A)$.

   SOLUTION: In the previous problem, we showed that if $\mathbf{x} \in \text{Null}(A^T A)$, then $\|A\mathbf{x}\| = 0$. This implies that $A\mathbf{x} = \mathbf{0}$, or equivalently, that $\mathbf{x} \in \text{Null}(A)$.

   Altogether, this problem is showing that the null spaces of A and $A^T A$ are the same!

5. Using the last problem, what can we conclude about the rank of A versus the rank of $A^T A$?

   SOLUTION: If A is $m \times n$, then the null spaces of A and $A^T A$ are the same subspaces of $\mathbb{R}^n$- thus they also have the same dimension. Therefore, the dimension of Row(A) and Row($A^T A$) are the same, and therefore, the dimension of Col(A) and Col($A^T A$) are the same. Therefore, A and $A^T A$ have the same rank.

6. Suppose I have a model equation: $y = \beta_0 + \beta_1 \sin(v) + \beta_2 \ln(w)$.

   Given the following data, set up the matrix equation from which we could determine a least squares solution for the β's:

$$\begin{array}{ccc} v & w & y \\ -1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 3 & -1 \\ 3 & 2 & 0 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 & \sin(-1) & \ln(2) \\ 1 & \sin(1) & \ln(1) \\ 1 & \sin(0) & \ln(3) \\ 1 & \sin(3) & \ln(2) \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$$

   Side Remark: In Matlab, you could solve this:

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v=[-1 1 0 3]'; w=[2 1 3 2]'; y=[1 2 -1 0]';
A=[ones(4,1), sin(v), log(w)];
beta=A\y;
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7. Given vectors $\mathbf{u}, \mathbf{v}$ in the vector space $\mathbb{R}^n$ with the usual dot product as inner product, show that the Pythagorean Theorem still holds. That is, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other, then:

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

SOLUTION: Write out the left side in terms of the dot product, and expand.

$$\|\mathbf{u}+\mathbf{v}\|^2 = (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}$$

Since $\mathbf{u}\cdot\mathbf{v} = 0$, this expression reduces to

$$\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

8. Orthogonally diagonalize the symmetric matrix $A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix}$.

SOLUTION: We notice that this matrix is symmetric. Find the eigenvalues first:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \quad \Rightarrow \quad \lambda^2 - 11\lambda + 24 = 0 \quad \Rightarrow \quad (\lambda-8)(\lambda-3) = 0$$

For $\lambda = 8$, we solve the following (I'm just using the first equation since the two equations should be constant multiples of each other):

$$(7-8)v_1 + 2v_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = 2v_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Similarly, for $\lambda = 3$:

$$(7-3)v_1 + 2v_2 = 0 \quad \Rightarrow \quad 4v_1 + 2v_2 = 0 \quad \Rightarrow \quad \begin{array}{ll} v_1 & = v_1 \\ v_2 & = -2v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Therefore, $A = PDP^T$, where

$$P = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix}$$

(NOTE: The order of the columns should correspond to the order of the eigenvalues!)

9. True or False, and explain: For every non-zero vector $\mathbf{v} \in \mathbb{R}^n$, the matrix $\mathbf{v}\mathbf{v}^T$ is called a projection matrix.

Generally, that would be false, but if $\mathbf{v}$ were a unit vector, then it would be true, since

$$\mathrm{Proj}_{\mathbf{v}}(\mathbf{x}) = \frac{\mathbf{v}^T\mathbf{x}}{\mathbf{v}^T\mathbf{v}}\mathbf{v} = \mathbf{v}(\mathbf{v}^T\mathbf{x}) = \mathbf{v}\mathbf{v}^T\mathbf{x}$$

10. Show that, if A is symmetric, then any two eigenvectors from distinct eigenvalues are orthogonal. Hint: Start with $\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2$, and see if you can transform this into $\lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2$.

    SOLUTION: Starting with the hint,

    $$\lambda_1 \mathbf{v}_1 \cdot \mathbf{v}_2 = (A\mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_1^T A^T \mathbf{v}_2 = \mathbf{v}_1^T (A\mathbf{v}_2) = \mathbf{v}_1^T \lambda_2 \mathbf{v}_2 = \lambda_2 \mathbf{v}_1 \cdot \mathbf{v}_2$$

    Subtracting the right side:

    $$(\lambda_1 - \lambda_2)\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$$

    Since $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1$ must be orthogonal to $\mathbf{v}_2$.

11. Suppose we have the matrix $A = [1, 1, 1]$.

    (a) What will the singular values of A be? (Try to compute them in the easiest possible way).

    SOLUTION: We can use either AA^T or A^TA. In this case, use $AA^T = 3$. The eigenvalue is 3, the others are 0. Therefore, there is one non-zero singular value $\sigma_1 = \sqrt{3}$.

    (b) Find (by hand) the reduced SVD for the matrix A. See if you can do it without any computation.

    SOLUTION: The reduced SVD would look like:

    $$[1, 1, 1] = 1 \cdot \sqrt{3} \cdot \begin{bmatrix} * \\ * \\ * \end{bmatrix}^T$$

    so we see that $\mathbf{u}_1 = 1$ and $\mathbf{v}_1 = \frac{1}{\sqrt{3}}[1, 1, 1]^T$.

    (c) Find a basis for the null space of A using the rest of the SVD that hasn't been computed yet (this one we'll need to compute).

    For the other two columns of V, we solve for the null space of [111], or:

    $$\begin{array}{ll} v_1 & = -v_2 - v_3 \\ v_2 & = v_2 \\ v_3 & = \quad\quad v_3 \end{array} \quad \Rightarrow \quad \mathbf{v} = v_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

    Side Remark: The full SVD for the problem would be:

    $$[1, 1, 1] = 1 \cdot [\sqrt{3}, 0, 0] \begin{bmatrix} 1/\sqrt{3} & -1\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1\sqrt{2} & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{2} \end{bmatrix}^T$$

12. Show that the eigenvalues of A^TA are non-negative. Hint: Consider $\|A\mathbf{v}_i\|$.

    SOLUTION: Using the hint, we have

    $$\|A\mathbf{v}_i\|^2 = (A\mathbf{v}_i)^T(A\mathbf{v}_i) = \mathbf{v}_i^T A^T A \mathbf{v}_i = \lambda_i \mathbf{v}_i^T \mathbf{v}_i = \lambda_i$$

Therefore, λ_i must be non-negative.

Side Remark: This was an important result so that we could define the singular values as the square root of these eigenvalues.

13. Suppose the SVD was given as the following:

$$A = \begin{bmatrix} 0.65 & -0.75 & 0 \\ 0 & 0 & 1 \\ 0.75 & 0.65 & 0 \end{bmatrix} \begin{bmatrix} 15.91 & 0 & 0 \\ 0 & 3.26 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -0.52 & -0.62 & -0.57 \\ -0.27 & 0.76 & -0.57 \\ -0.80 & 0.14 & 0.57 \end{bmatrix}^T$$

(a) What is the rank of A? SOLUTION: The rank is the number of non-zero singular values, so in this case, the rank is 2.

(b) Write a basis for the column space and null space of A.

SOLUTION: The column space is spanned by the first two columns of U, and the null space is spanned by the last column of V.

(c) Write the matrix product for the pseudoinverse of A (you don't need to multiply it out).

SOLUTION: Symbolically (Matlab notation for the columns), it is $V(:, 1:2)\Sigma^{-1}(1:2, 1:2)U(:, 1:2)$

$$\begin{bmatrix} -0.52 & -0.62 \\ -0.27 & 0.76 \\ -0.80 & 0.14 \end{bmatrix} \begin{bmatrix} \frac{1}{15.91} & 0 \\ 0 & \frac{1}{3.26} \end{bmatrix} \begin{bmatrix} 0.65 & -0.75 \\ 0 & 0 \\ 0.75 & 0.65 \end{bmatrix}$$

14. Suppose A is square and invertible. Find the SVD of A^{-1}.

SOLUTION: If A is square and invertible, then in the SVD for A:

$$A = U\Sigma V^T$$

the matrices U and V are orthogonal (so $UU^T = U^TU = I$, and similarly for V):

$$A^{-1} = V \begin{bmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & 0\cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_n} \end{bmatrix} U^T$$

15. Show that if A is square, then $|\det(A)|$ is the product of the singular values of A.

SOLUTION:

$$A = U\Sigma V^T \quad \Rightarrow \quad \det(A) = \det(U\Sigma V^T)$$

Here is where we need all matrices to be square- So the determinant is defined:

$$\det(A) = \det(U)\det(\Sigma)\det(V)$$

Since U and V are orthogonal, each of their determinants is ± 1 (be sure that you can prove this). Therefore,

$$\det(A) = \pm \det(\Sigma) = \pm \sigma_1 \sigma_2 \cdots \sigma_n$$