4.1, 32 If H, K are subspaces, show that $H \cap K$ is also a subspace.

- Show that $H \cap K$ has the zero vector:
  Since H, K are subspaces, then $\vec{0} \in H$ and $\vec{0} \in K$, so $\vec{0} \in H \cap K$.
- Show that $H \cap K$ is closed under addition:
  Let $\mathbf{x}_1, \mathbf{x}_2$ be in $H \cap K$. That means that each is in H and each is in K. Since both vectors are in H (and H is a subspace), then $\mathbf{x}_1 + \mathbf{x}_2$ is in H. Similarly, since both vectors are in K (and K is a subspace), then $\mathbf{x}_2 + \mathbf{x}_2$ is in K. Therefore, the sum is also in $H \cap K$.
- Show that $H \cap K$ is closed under scalar multiplication:
  Let $\mathbf{x} \in H \cap K$. We show that $c\mathbf{x}$ is in $H \cap K$ for all scalars c:
  Since $\mathbf{x} \in H \cap K$, then $\mathbf{x} \in H$ and $\mathbf{x} \in K$. Since each of these are subspaces, $c\mathbf{x} \in H$ and $c\mathbf{x} \in K$. Therefore, $c\mathbf{x} \in H \cap K$.

The argument above does not work for unions-For example,

$$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \quad K = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

Then $H \cap K = \left\{ \vec{0} \right\}$, but the union (only being the two coordinate axes) would not be closed under addition- Adding the two basis vectors together, in fact, gives $(1, 1)$, which is not in H or in K.

4.2, 30 Let $T : V \to W$ be a linear transformation from vector space V to vector space W. Show that the range of T is a subspace of W.

NOTE: To show that something is in the range of T, we must show that it came from some vector in V...

- Is $\vec{0}$ in the range of T? Yes, since $T(\vec{0}) = \vec{0}$, and V is a subspace (so the domain contains the zero vector).
- Is the range closed under addition?
  Let $\mathbf{w}_1, \mathbf{w}_2$ be in the range of T. Then there are vectors $\mathbf{v}_1, \mathbf{v}_2$ in V such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Now,

  $$\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2)$$

  And, since V is a vector space, the sum $\mathbf{v}_1 + \mathbf{v}_2$ is in V. Therefore, W is closed under addition.
- Is the range closed under scalar multiplication?
  Let $\mathbf{w}$ be in the range, so $\mathbf{w} = T(\mathbf{v})$, for some $\mathbf{v} \in V$. Since V is a subspace, then $c\mathbf{v}$ is in V for all scalars c, which tells us:

  $$c\mathbf{w} = cT(\mathbf{v}) = T(c\mathbf{v})$$

  And W is closed under scalar multiplication.