Section 2.3: Modeling

1. The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the area initially and predators eat 20,000 mosquitoes/day. Determine the population of mosquitoes in the area at any time.

   SOLUTION: Using the absence of any other factors, we have exponential growth, $P' = kP$. We can determine k using the doubling time- Since $P(t) = P_0 e^{kt}$, then $2P_0 = P_0 e^{7k}$ (if time is measured in days). Solving for k leads to

   $$k = \ln(2)/7 \approx 0.099$$

   Now changing the model to accommodate the predators, we'll have the following model, where t is in days:

   $$P' = kP - 20,000$$

   Putting k back in, the solution is

   $$P(t) = Ce^{kt} + \frac{20,000}{k} = Ce^{0.099\,t} + 202020.2 = -2020.2e^{0.099\,t} + 202020.2$$

2. A pond initially contains 500,000 gallons of unpolluted water has an outlet that releases 10,000 gallons of water per day. A stream flows into the pond at 12,000 gallons per day containing water with a concentration of 2 grams per gallon of a pollutant. Find a differential equation that models this process and determine what the concentration of pollutant will be after 10 days.

   SOLUTION: Total Rate = Rate In − Rate Out, where rate is in grams per day. The incoming rate is $12,000 \times 2$ grams per day. The outgoing rate is a little harder, since more water is coming in than is being pumped out- In fact, we're gaining 2,000 gallons per day. Therefore, if $Q(t)$ is the grams of pollutant in the tank at t days, then the rate out:

   $$\frac{Q \text{ grams}}{500,000 + 2,000t \text{ gallons}} \cdot \frac{10,000 \text{ gallons}}{1 \text{ day}} = \frac{10Q}{500 + 2t}$$

   Now the IVP: $Q' = 24,000 - \frac{10Q}{500+2t}$, with $Q(0) = 0$. Solving this, you should find that the integrating factor is

   $$e^{\int p(t)\,dt} = (500 + 2t)^5$$

   Then we should find that

   $$Q(t) = 2000(500 + 2t) + \frac{C}{(500 + 2t)^5}$$

   The value of C is very big in this case- about -3^{19}. Using a calculator/computer, we should find that after 10 days we have approximately 218000 grams (approximated because of C).

3. An object with temperature 150°F is placed in a freezer whose temperatureis 30°F. Assume that the temperature of the freezer remains essentially constant. If the object is cooled to 120°F after 8 min, what will its temperature be after 18 min?

   Using Newton's law of cooling, $u' = -k(u - T)$, so that $u(t) = C\mathrm{e}^{-kt} + T$. Plugging things in, we get
$$u(t) = C\mathrm{e}^{-kt} + 30$$
   We have two unknowns, so we need two data points- We're given $(0, 150)$ and $(8, 120)$ (with time in minutes). Using the first data point, we can solve for C ($C = 120$), then with the second data point, we have
$$120 = 120\mathrm{e}^{-8k} + 30 \quad \Rightarrow \quad k \approx 0.036$$
   Finally, we evaluate our temperature function at $t = 18$:
$$u(18) = 120\mathrm{e}^{-0.036\times 18} + 30 \approx 62.77 + 30 = 92.77$$

4. You just won the lottery. You put your $5,000,000 in winnings into a fund that has a rate of return of 4%. Each year you use $300,000. How much money will you have twenty years from now?

   Assuming continuous compounding of interest (so that we have a DE), if $S(t)$ is the amount of money in the fund at time t (in years), then
$$S' = 0.04S - 300,000 \quad \Rightarrow \quad S(t) = C\mathrm{e}^{0.04\,t} + \frac{300,000}{0.04} = C\mathrm{e}^{0.04\,t} + 7,500,000$$
   To finish, find C using the initial condition $S(0) = 5,000,000$, and then substitute $t = 20$ into the expression to get $S(20) \approx 1,936,148$.

5. A bottle of orange juice being taken out of refrigerator at $2°C$ warms up to $5^{\mathrm{circ}}C$ in 5 minutes while sitting in a room of temperature $23°C$. How warm will the orange juice be if left out for 15 min?

   SOLUTION: Use Newton's Law of Cooling:
$$u' = -k(u - T) \quad \Rightarrow \quad u(t) = C\mathrm{e}^{-kt} + T = C\mathrm{e}^{-kt} + 23.$$
   We need two data points to solve for the two constants. At time 0, temp is 2, and 5 minutes later, 5. Given that, we should find (t in minutes)
$$u(t) = -21\mathrm{e}^{-0.04\,t} + 23$$
   After 15 minutes, the orange juice is approximately 10.9 deg C.

6. A 50 kg object is shot from a cannon "straight up" with an initial velocity of 10 m/s off a bridge that is 100 meters above the ground. If air resistance is estimated as $5v(t)$ (with appropriate sign) determine the velocity of the mass when it hits the ground (you may assume the object does not hit the bridge on the way down).

   SOLUTION: Our model was (using $v(t)$ as velocity)

$$mv' = -mg + \gamma v \quad \Rightarrow \quad v' = \frac{\gamma}{m}v - g, \quad v(0) = 10$$

   Using the solution for $y' = ay + b$, which is $Ce^{at} - b/a$, the solution is

$$v(t) = Ce^{\gamma t/m} + \frac{mg}{\gamma} = Ce^{t/10} + 98 \quad \Rightarrow \quad v(t) = -88e^{t/10} + 98$$

   Now that we've completely determined the velocity, we can compute the position by antidifferentiation.

$$s(t) = -880te^{t/10} + 98t + C \text{ with } s(0) = 100 \quad \Rightarrow \quad C = 980$$

   Now we would need to solve for where $s(t) = 0$ (this would have to be done numerically). Doing so, we see t is very close to 5, then substitute this number into $v(t)$ to find the velocity at that time.

7. Initially a tank contains 10000 litres of brine with a salt concentration of 1 kg salt per 100 litres. Brine with 2 kg salt per 100 litres enters the tank at a rate of 20 litres per second. The well-stirred mixture leaves at the same rate. Find the concentration of salt as a function of time.

   TYPO: Let's determine the amount of salt at time t rather than the concentration.

   SOLUTION: Our usual "Rate in−Rate out", so let $Q(t)$ be the kg of salt at time t.

$$\frac{dQ}{dt} = \frac{2}{100} \cdot \frac{20}{1} - \frac{Q}{10000}\frac{20}{1}$$

   Simplifying a bit and using the initial condition $Q(0) = 100$, we get that

$$Q(t) = -100e^{-t/500} + 200$$

   (a) Take the same setup as in the previous example, but the mixture leaves the tank at only 10 litres per second. Of course the tank will eventually fill up, but we want to know the amount of salt at any time before this.

$$S' = \frac{2}{5} - \frac{10}{10,000 + 10t}Q$$

   Simplify this a bit before solving it as a linear DE.

8. (You might use a calculator for this one)

   The temp of a glass of iced tea is initially 5 deg. (all in C). After 5 minutes the tea has heated to 10 deg. in a room where the air temp is 30 deg.

   SOLUTION: This is Newton's law of cooling, $u' = -k(u - T)$, or $u(t) = C\mathrm{e}^{-kt} + T$.

   We have two data points like the previous law of cooling problem from which we determine the values of k and C.

   (a) Determine the temp of tea as a function of t.

   $$u(t) = -25\mathrm{e}^{-0.094\,t} + 30$$

   (b) What is the temp of the tea after 10 minutes?
   SOLUTION: Just substitute $t = 10$ into $u(t)$ and evaluate.

   (c) When will the tea reach a temp of 20 deg?
   Solve for t: $20 = -25\mathrm{e}^{-0.094\,t} + 30$

9. A home buyer can afford to spend no more than \$800/month on mortgage payments. Suppose that the interest rate is 9% and that the term of the mortgage is 20 years. Assumed that interest is compounded continuously and that payments are also made continuously. Determine the maximum amount that this buyer can afford to borrow.

   SOLUTION: If $S(t)$ is the amount we owe on the loan, then the amount we borrow is $S(0) = S_0$. The model we're using is:

   $$S' = 0.09S - (800)(12) = -0.09S + 9600$$

   We can solve this as $S(t) = C\mathrm{e}^{0.09\,t} + 106,666.67$. We want this amount to be zero at $t = 20$, so we can solve for C from that. Once C is computed, then compute $S(0)$.

10. A tank initially contains 100 L of fresh water. A brine containing 200 g/L of salt salt flows into the tank at rate of 3 L/min. The solution inside the tank is kept well stirred and flows out of the tank at the rate 2 L/min. Determine the concentration of salt at any time.

    SOLUTION: Note that, if $Q(t)$ is the amount (in grams) of salt at time t, then the concentration of salt in the tank would be given by

    $$\frac{Q(t)}{100 + t}$$

    So, let's determine the usual- The grams of salt in the tank at time t.

    $$Q' = 600 - \frac{2}{100 + t} \cdot Q, \qquad Q(0) = 0$$

Then solve as a linear DE- The integrating factor is (put in standard form first)

$$e^{\int \frac{2}{100+t}\,dt} = e^{2\ln|100+t|} = (100+t)^2$$

Now,

$$(Q(100+t)^2)' = 600(100+t)^2 \quad \Rightarrow \quad Q(t)(100+t)^2 = 200(100+t)^3 + C$$

$$Q(t) = 200(100+t) + \frac{C}{(100+t)^2}$$

Then solve for C (it's kind of messy in this model).

11. We want to use the logistic equation to model the number of people sick with the flu. We have a total population of 1000, and on the first day, one person was sick. On day 5 there were 20 people sick. We want to find the appropriate constants and solve the DE.

    HINT: For a short amount of time, we can get a fairly good model by using exponential growth.

    SOLUTION: There was a typo in the question- Should be that on day 5, there were 20 sick. For simplicity, let t be the number of days since the first day (so the first day is $t = 0$). Then in the short term,

    $$P' = kP \quad \Rightarrow \quad P(t) = e^{kt}$$

    Using the data point $(4, 20)$, we find that $k = \ln(20)/4 \approx 0.75$. Now that k has been determined, we can go back to the full logistic model with carrying capacity of 1000:

    $$P' = 0.75\,P\left(1 - \frac{P}{1000}\right)$$

    The integral gets a bit messy, but let's go into it a little to be sure we can...

    $$\frac{1}{P(1-P/1000)}\,dP = 0.75\,dt \quad \Rightarrow \quad \int \frac{1000}{P(1-P)}\,dP = 0.75t + C$$

    (and so on).

12. A brine solution of salt flows at a constant rate of 4 L/min into a large tank that initially held 100 L of pure water. The solution inside the tank is kept well stirred and flows out of the tank at 2 L/min. If the concentration of salt in the brine entering the tank is 0.2 kg/L, determine the mass of salt in the tank after t minutes.

    SOLUTION: Here's the IVP. Note that we're gaining 2 L/min, so the amount of water at time t would be $100 + 2t$.

    $$\frac{dQ}{dt} = 0.2 \cdot 4 - \frac{2Q}{100+2t} \qquad Q(0) = 0$$

13. (Sorry for the gruesome problem!)

Suppose a dead body is discovered at 3:45PM in a room where the temperature is 20 deg C. At the time of discovery, the temperature of the body is 27 deg C. Two hours later (at 5:45PM), the temperature of the body is 25.3 degrees. What was the time of death?

Note that the normal adult human has a body temp of approx 37 deg C.

SOLUTION: Here's the IVP. For Newton's law of cooling, $u' = -k(u - T)$. In this case, let t be the number of hours past 3:45PM.

$$\frac{du}{dt} = -k(u - 20), \quad u(0) = 27$$

Solving, we get

$$u(t) = 7\mathrm{e}^{-kt} + 20$$

We're told the curve should go through $(2, 25.3)$, so from that we can determine k:

$$25.3 = 7\mathrm{e}^{-2k} + 20 \quad \Rightarrow \quad k = -ln(0.757)/2 \approx 0.139$$

For time of death, take

$$37 = 7\mathrm{e}^{-0.139\,t} + 20$$

We should find time of death to be approximately 6 hours, 23 minutes prior to discovery.