Sample Questions (Chapter 3, Math 244)

1. True or False?

   (a) The characteristic equation for $y'' + y' + y = 1$ is $r^2 + r + 1 = 1$

   (b) The characteristic equation for $y'' + xy' + e^x y = 0$ is $r^2 + xr + e^x = 0$

   (c) The function $y = 0$ is always a solution to a second order linear homogeneous differential equation.

   (d) In using the Method of Undetermined Coefficients, the ansatz $y_p = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x))$ is equivalent to
   $$y_p = (Ax^2 + Bx + C)\sin(x) + (Dx^2 + Ex + F)\cos(x)$$

   (e) The operator $T(y) = y' + t^2 y + 1$ is a linear operator (in y).

2. (a) First, solve the DE: $y'' + 4y' + 3y = 0$. (b) Use Cramer's rule to find the constants, if the initial conditions are $y(0) = 1$, $y'(0) = \alpha$.

3. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

4. What do we need to check in order to see if two functions, $y_1.y_2$ form a **fundamental set of solutions** to a given second order linear homogeneous DE?

5. If $W(f, g) = t^2 e^t$ and $f(t) = t$, find $g(t)$.

6. If y_1, y_2 form a fundamental set of solutions to: $t^2 y'' - 2y' + (3 + t)y = 0$, and if $W(y_1, y_2)(2) = 3$, find $W(y_1, y_2)(4)$.

7. Given that $y_1 = \frac{1}{t}$ solves the differential equation: $t^2 y'' - 2y = 0$, find a fundamental set of solutions using Abel's Theorem.

8. Give the general solution:

   (a) $y'' - 3y' - 10y = 0$ (b) $y'' + 4y' + 4y = 0$ (c) $y'' - 4y' + 5y = 0$

9. Suppose the roots to the characteristic equation are as given below. Write the general solution to the DE, and write down what the second order linear homogeneous DE was.

   (a) $r = -2, 3$ (b) $r = 1, 1$ (c) $r = 2 \pm 3i$

10. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $e^{(3-2i)t}$ (iii) $e^{i\pi}$

11. Write $a + ib$ in polar form: (i) $-1 - \sqrt{3}i$ (ii) $3i$ (iii) -4 (iv) $\sqrt{3} - i$

12. Write each expression as $R\cos(\omega t - \delta)$

    (a) $3\cos(2t) + 4\sin(2t)$ (b) $-\cos(t) + \sqrt{3}\sin(t)$ (c) $4\cos(3t) - 2\sin(3t)$

13. Practice setting up the Spring-Mass model: If you need it, $g \approx 9.8 = \frac{49}{5}$.

    (a) Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

(b) A mass of 0.5 kg stretches a spring an additional 0.05 meters to get to equilibrium. (i) Find the spring constant. (ii) Does a stiff spring have a large spring constant or a small spring constant (explain).

(c) It takes 6 N of force to stretch a certain spring 3 meters. A mass of $\frac{1}{2}$ kg is attached to a spring. (a) Find the spring constant, and (b) if the damping constant is 2, write the differential equation for the motion of the mass.

(d) Given the model of motion of the mass on a spring is given by

$$\frac{1}{2}u'' + \gamma u' + 8u = 0$$

Find γ so that the spring is *underdamped*, *critically damped*, *overdamped.*

14. Solve. If there are initial conditions, solve for all constants, otherwise, find the general solution.

    (a) $u'' + u = 3t + 4$, $u(0) = 0, u'(0) = 0$.
    (b) $u'' + u = \cos(2t)$, $u(0) = 0$, $u'(0) = 0$
    (c) $u'' + u = \cos(t)$, $u(0) = 0$, $u'(0) = 0$ (And please compare to the previous problem).
    (d) $y'' + 4y' + 4y = e^{-2t}$
    (e) $y'' - 2y' + y = te^{t} + 4$, $y(0) = 1$, $y'(0) = 1$.

15. Find a second order linear differential equation with constant coefficients whose general solution is given by:

$$y(t) = C_1 + C_2 e^{-t} + \frac{1}{2}t^2 - t$$

16. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

17. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.
    If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

18. For each DE below, use the Method of Undetermined Coefficients to give the final form of your guess for the particular solution, $y_p(t)$. Do NOT solve for the coefficients.

    (a) $y'' + 3y' = t^3 + t^2 e^{-t} + \sin(3t)$
    (b) $y'' + y = t(1 + \sin(t))$
    (c) $y'' - 5y' + 6y = e^{2t}(3t + 4)\sin(t)$
    (d) $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos(t) + 4e^{-t}t^2\sin(t)$

19. Each equation below exhibits either **beating**, **resonance**, or **neither**. Label each one with B, R, or N.

    (a) $y'' + 3y = \cos(3t)$
    (b) $y'' + 9y = \cos(3t)$
    (c) $y'' + (1.1)^2 y = \sin(t)$
    (d) $y'' + 3y = \cos(9t)$