Summary- Elements of Chapters 7

Systems and Conversions

If we have the generic system of autonomous differential equations:

$$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

We might be able to solve the "unparameterized" DE: $\dfrac{dy}{dx} = \dfrac{g(x,y)}{f(x,y)}$.

Looking at the linear first order system, we learned how to convert it to an equivalent second order differential equation, and alternatively, we can convert a second (or higher) differential equation into a system of first order.

Eigenvalues and Eigenvectors

For the following, we are solving the system:

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} x' \\ y' \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \Leftrightarrow \quad \mathbf{x}' = A\mathbf{x}$$

1. **Main Definition**: If there is a constant λ and a non-zero vector $\mathbf{v}$ that solves

   $$\begin{aligned} av_1 \quad +bv_2 &= \lambda v_1 \\ cv_1 \quad +dv_2 &= \lambda v_2 \end{aligned}$$

   then λ is an **eigenvalue**, and $\mathbf{v}$ is an associated **eigenvector**.

2. To solve for the eigenvalues, note the logical progression:

   $$\begin{aligned} av_1 \quad +bv_2 &= \lambda v_1 \\ cv_1 \quad +dv_2 &= \lambda v_2 \end{aligned} \Leftrightarrow \begin{aligned} (a-\lambda)v_1 \quad +bv_2 &= 0 \\ cv_1 \quad +(d-\lambda)v_2 &= 0 \end{aligned} \tag{1}$$

   This system has a non-zero solution for v_1, v_2 only if the two lines are multiples of each other. In that case, the determinant must be zero.

   $$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad \Rightarrow \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

   And this is the **characteristic equation**. This is formallly solved via the quadratic formula, but we would typically factor it or complete the square. For each λ, we must go back and solve Equation (1) to find $\mathbf{v}$. For example, if we have the line on the left, the eigenvector can be written down directly (as long as the equation is not $0 = 0$)

   $$(a-\lambda)v_1 + cv_2 = 0 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -c \\ a-\lambda \end{bmatrix}$$

Solve $\mathbf{x}' = A\mathbf{x}$

1. We make the ansatz: $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$, substitute into the DE, and we find that $\lambda, \mathbf{v}$ must be an eigenvalue, eigenvector of the matrix A.

2. The eigenvalues are found by solving the characteristic equation:

   $$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

   The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 with two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Re}\left(e^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Im}\left(e^{\lambda t}\mathbf{v}\right)$$

- One eigenvalue, one eigenvector (which is not needed). Determine $\mathbf{w}$, where:

$$\begin{aligned} (a-\lambda)x_0 + cy_0 &= w_1 \\ cx_0 + (d-\lambda)y_0 &= w_2 \end{aligned}$$

  Then

$$\mathbf{x}(t) = e^{\lambda t}\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}\right) = e^{\lambda t}(\mathbf{x}_0 + t\mathbf{w})$$

  Note: In this solution, we don't have arbitrary constants- rather, we're writing the solution in terms of the initial conditions.

You might find this helpful- Below there is a chart comparing the solutions from Chapter 3 to the solutions in Chapter 7:

	Chapter 3	Chapter 7
Form:	$ay'' + by' + cy = 0$	$\mathbf{x}' = A\mathbf{x}$
Ansatz:	$y = e^{rt}$	$\mathbf{x} = e^{\lambda t}\mathbf{v}$
Char Eqn:	$ar^2 + br + c = 0$	$\det(A - \lambda I) = 0$
Real Solns	$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$	$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
Complex	$y = C_1 \mathrm{Re}(e^{rt}) + C_2 \mathrm{Im}(e^{rt})$	$\mathbf{x}(t) = C_1 \mathrm{Re}\left(e^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Im}\left(e^{\lambda t}\mathbf{v}\right)$
SingleRoot	$y = e^{rt}(C_1 + C_2 t)$	$\mathbf{x}(t) = e^{\lambda t}\left(\mathbf{x}_0 + t\mathbf{w}\right)$

Classification of the Equilibria

The origin is always an equilibrium solution to $\mathbf{x}' = A\mathbf{x}$, and we can use the Poincaré Diagram to help us classify the origin by using the trace, determinant and the discriminant.