Solutions to Final Exam Sample 1

1. Short Answer:

   (a) True or False (give a short reason): The solution to $y' = f(y)$ may be periodic.
   SOLUTION: False. If a DE is autonomous, that means that the local tangent line slopes must not change over time. Therefore, if a function is increasing, it cannot decrease.

   (b) Evaluate the integral: $\displaystyle\int_0^5 \delta(t-2) + 3u_4(t)\,dt$
   SOLUTION: The integral of the delta function is 1, and $u_4(t)$ will build a rectangle of height 1 from time 4 to time 5. The answer is $1 + 3 = 4$.

   (c) Given the system below, convert to a second order DE (do not solve the DE):
   $$\begin{aligned} x_1' &= 2x_1 + x_2 \\ x_2' &= x_1 + 2x_2 \end{aligned}$$
   SOLUTION: You can write this in terms of x_1 or x_2 (you'll get the same DE).
   $$x_1'' - 4x_1' + 3x_1 = 0$$

2. Short Answer

   (a) Solve: $y'' + 4y' + 4y = 0$. Solution: $y(t) = \mathrm{e}^{-2t}(C_1 + C_2 t)$

   (b) Solve: $y'' + 2y' + 5y = 0$. Solution: $y(t) = \mathrm{e}^{-t}(C_1 \cos(2t) + C_2 \sin(2t))$

   (c) Label each of the previous two problems as: Underdamped, Overdamped, or Critically Damped.
   SOLUTION: The first was critically damped, the second was underdamped.

3. Give your (final) ansatz for the particular solution to the following using the Method of Undetermined Coefficients. Do NOT actually find the coefficients!

   (a) $y' - 5y = t^2$
   SOLUTION: $y_p(t) = At^2 + Bt + C$.

   (b) $y'' - 2y' + 5y = t\mathrm{e}^t \sin(2t)$
   SOLUTION: $y_p(t) = t\mathrm{e}^t((At + B)\cos(2t) + (Ct + D)\sin(2t))$

4. Consider the system: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ \alpha & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

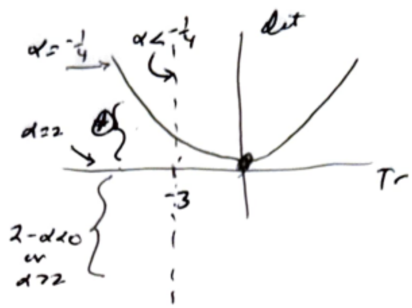
   Using the Poincaré Diagram (you might sketch it), describe how changing α changes the classification of the origin.

   From our analysis below, we see that, if $\alpha < -1/4$, the origin is a spiral sink. If $\alpha = -1/4$, the origin is a degenerate sink, if $-1/4 < \alpha < 2$, the origin is a sink, and if $\alpha = 2$, we have a line of stable fixed points. If $\alpha > 2$, the origin is a saddle.

$Tr(A) = -3$

$\det(A) = 2-\alpha$

$-\frac{1}{4} < \alpha < 2$



$\Delta = 9 - 4(2-\alpha)$

$= 9 - 8 + 4\alpha = 1 + 4\alpha < 0$

$4\alpha < -1$

$\alpha < -1/4$

5. Find the eigenvalues and eigenvectors for the matrix $\begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$

   SOLUTION: $\lambda = 5, \mathbf{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\lambda = 1, \mathbf{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$

6. Consider the system: $\begin{array}{l} x' = 2x + 6y \\ y' = 2x - 2y \end{array}$

   (a) Find the general solution.

   $$C_1 e^{4t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{-4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

   (b) Find the particular solution with initial value $(1, 3)$.
   SOLUTION: Several ways of computing this. We should find that $C_1 = 1$ and $C_2 = 2$.

7. Laplace Transforms:

   (a) Find the Laplace transform of the solution, $Y(s)$ (do **not** solve for $y(t)$!)

   $$y'' - 2y' + y = g(t) \qquad y(0) = 1, \quad y'(0) = 2, \qquad \text{where } g(t) = \begin{cases} t/2 & \text{if } 0 \le t < 2 \\ 1 & \text{if } t \ge 2 \end{cases}$$

   SOLUTION: First write $g(t) = t/2(u_0(t) - u_2(t)) + u_2(t) = \frac{t}{2} + (1 - \frac{t}{2})u_2(t)$.
   Next, take the Laplace transform of both sides:

   $$Y(s) = \frac{1 - e^{-2s}}{2s^2(s^2 - 2s + 1)} + \frac{s}{s^2 - 2s + 1}$$

(b) Compute the inverse Laplace transform of $\dfrac{\mathrm{e}^{-3s}(s+1)}{s^2+4s+6}$

SOLUTION: $y(t) = u_3(t)h(t-3)$, where

$$h(t) = \mathrm{e}^{-2t}\left(\cos(\sqrt{2}\,t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t)\right)$$

8. A tank originally contains 100 gallons of fresh water. Water containing 3/5 lb of salt per gallon is poured into the tank at a rate of 5 gal/min, and the mixture is pumped out at 7 gal/min. Write an IVP modeling the amount of salt at time t (do not solve it!).

$$\frac{dQ}{dt} = 3 - \frac{7}{100-2t}Q, \qquad Q(0) = 0$$

9. Find the general solution: $\dfrac{dy}{dt} - \dfrac{2}{t}y = t^3\mathrm{e}^t$

SOLUTION: $y(t) = (t^3 - t^2)\mathrm{e}^t + Ct^2$

10. Find the radius of convergence for the power series: $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty} \frac{|x|^{2(n+1)+1}}{|x|^{2n+1}} \cdot \frac{(2n+1)!}{(2(n+1)+1)!} = \lim_{n\to\infty} \frac{|x|^{2n+3}}{|x|^{2n+1}} \frac{1}{(2n+2)(2n+3)} = \lim_{n\to\infty} \frac{|x|^2}{(2n+2)(2n+3)} = 0$$

Therefore, the series converges no matter what x we choose (the radius is infinite).

11. Write the power series of the solution, $y(x)$, centered at $x_0 = 0$, up to and including the x^5 term (use the derivatives of y directly).

$$y'' - 2x^2y' + y = 0 \qquad y(0) = 1, \quad y'(0) = 1$$

SOLUTION:

$$y(x) = 1 + x - \frac{1}{2}x^2 - \frac{1}{3!}x^3 + \frac{5}{4!}x^4 - \frac{11}{5!}x^5 + \cdots$$