M244
Sample 02

Final Exam

Name:__________

Show all your work! You may not use your text, colleagues or a calculator. A table of Laplace transforms and the Poincare Diagram is provided.

1. Solve $\mathbf{x}' = A\mathbf{x}$ for the given matrix A:

   (a) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$. SOLUTION: $C_1 e^{3t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}$ SOLUTION: $e^{-t}\left[C_1\begin{bmatrix} \cos(t) \\ 2\cos(t)+\sin(t) \end{bmatrix} + C_2\begin{bmatrix} \sin(t) \\ -\cos(t)+2\sin(t) \end{bmatrix}\right]$

2. Convert the following system to an equivalent 2d order DE:

$$\mathbf{x}' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}\mathbf{x} \quad \Rightarrow \quad x_1'' + 2x_1' + 5x_1 = 0$$

3. Convert the following 2d order DE to an equivalent system of first order equations (do not solve): $y'' + 3y' + 5y = 0$

   SOLUTION: Let $x_1 = y, x_2 = y'$. Then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -5x_1 - 3x_2 \end{aligned}$$

4. Find the radius of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n(n+2)}$

$$\lim_{n\to\infty} \frac{n(n+2)}{(n+1)(n+3)}|x| = |x| < 1$$

   The radius of convergence is 1.

5. Find the recurrence relation between the coefficients of the power series solution to the following at the given value of x_0:

$$xy'' + y' + xy = 0, \qquad x_0 = 1$$

   (This was exercise 8, section 5.2- It ended up being more complicated than usual because we aren't able to start the series for the y'' term at 0, so we're forced to advance the sum in all the other terms to 1. The problem on the exam won't be this involved, so if you were able to get through this one, you're doing extra great! I've included some solutions on the last page if you really want to see what's going on).

6. Solve the following for $Y(s)$ (the Laplace transform of $y(t)$), but do NOT invert!

$$y'' + 2y' + 2y = \cos(3t) + \delta(t-2) \qquad y(0) = 1, y'(0) = 2$$

$$Y(s) = \frac{s}{(s^2+9)(s^2+2s+2)} + \frac{\mathrm{e}^{-2s}}{s^2+2s+2} + \frac{s+4}{s^2+2s+2}$$

7. Invert the Laplace transform: $\dfrac{\mathrm{e}^{-2s} - \mathrm{e}^{-3s}}{2s^2+s+2}$

SOLUTION: Write this as $(\mathrm{e}^{-2s} - \mathrm{e}^{-3s})H(s)$, then the solution is

$$u_2(t)h(t-2) - u_3(t)h(t-3), \qquad h(t) = \frac{2}{\sqrt{15}}\mathrm{e}^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

8. Guess the final form of the particular solution for each DE using the Method of Undetermined Coefficients (do NOT solve for the coefficients)

(a) $y'' + 3y' = 2t^4 + t^2\mathrm{e}^{-3t} + \sin(3t)$

SOLUTION: The coefficients for each part are independent of each other.

$$y_{p_1}(t) = (At^4 + Bt^3 + Ct^2 + Dt + E)t$$

$$y_{p_2}(t) = (At^2 + Bt + C)\mathrm{e}^{-3t} \cdot t$$

$$y_{p_3}(t) = A\cos(3t) + B\sin(3t)$$

(b) $y'' + y = t(1 + \sin(t))$

$$y_{p_1}(t) = At + B$$

$$y_{p_2}(t) = [(At+B)\cos(t) + (Ct+D)\sin(t)] \cdot t$$

9. If y_1, y_2 are a fundamental set of solutions to $t^2y'' - 2y' + (3+t)y = 0$, and if $W(y_1,y_2)(2) = 3$, then find the value of $W(y_1,y_2)(4)$.

In the Wronskian, we compute $C\mathrm{e}^{-\int -2/t^2\,dt} = C\mathrm{e}^{-2/t}$

$$W(y_1,y_2)(2) = 3 \quad \Rightarrow \quad C\mathrm{e}^{-2/2} = 3 \quad \Rightarrow \quad C = 3e$$

Now,

$$W(y_1,y_2)(4) = 3e\mathrm{e}^{-2/4} = 3\sqrt{e}$$

10. A spring with a mass of 3 kg is held stretched 0.6 m beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s, find the IVP modeling the position of the mass at time t (assume no damping). Do NOT solve.

    SOLUTION: We need the spring constant. Since the restorative force is proportional to the length past natural length, we have

$$20 = k \cdot \frac{6}{10} = k \cdot \frac{3}{5} \quad \Rightarrow \quad k = 20 \cdot \frac{5}{3} = \frac{100}{3}$$

    Therefore, $3y'' + \frac{100}{3}y = 0$, $y(0) = 0$, $y'(0) = 1.2$

11. A tank initially contains 10 kg of salt in a tank that contains 100 gallons of brine. Brine is flowing into the tank at a rate of 5 gallons per minute, and contains 2 kg of salt per gallon. The well mixed brine is pumped out of the tank at a rate of 6 gallons per minute. Write the IVP that gives the amount of salt in the tank at time t (do NOT solve).

$$\frac{dQ}{dt} = 10 - \frac{6}{100 - t}Q \qquad Q(0) = 10$$

12. Solve:

    (a) $(9x^2 + y - 1)\,dx - (4y - x)dy = 0$ with $y(1) = 3$.
    SOLUTION: Exact, with $M_y = N_x = 1$. Integrating appropriately, we get

$$3x^3 - x + xy - 2y^2 = -13$$

    (b) $y' = 2x/(y + x^2y)$
    SOLUTION: Separable (denominator factors as $y(1 + x^2)$, so that

$$\frac{1}{2}y^2 = \ln(1 + x^2) + C$$

    (c) $ty' - y = t^2 \mathrm{e}^{-t}$, $t > 0$.
    SOLUTION: Linear. $y(t) = -t\mathrm{e}^{-t} + Ct$

13. Consider the system: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

    Using the Poincaré Diagram (you might sketch it), describe how changing α changes the classification of the origin.

    SOLUTION: The easiest way to consider this is to see where the trace, determinant and discriminant change sign. In this case, the trace is $\alpha + 2$, the determinant is $2\alpha + 2$

and the discriminant is an irreducible quadratic (always positive).

$\alpha + 2$	$-$	$+$	$+$
$2\alpha + 2$	$-$	$-$	$+$
Δ	$+$	$+$	$+$
	$\alpha < -2$	$-2 < \alpha < -1$	$\alpha > -1$

So, if $\alpha < -1$, the origin is a saddle. At $\alpha = -1$, we get a line of unstable fixed points, and for $\alpha > -1$, we have a source.

(a) For this equation $x_0 = 1$ is an ordinary point so we look for a soultion in the form of a power series about x_0,

$$y = \sum_{n=0}^{+\infty} a_n (x-1)^n,$$

which converges in some interval $|x| < \rho$. The series for y' and y'' are given by

$$y' = \sum_{n=0}^{+\infty} (n+1) a_{n+1} (x-1)^n$$

and

$$y'' = \sum_{n=0}^{+\infty} (n+2)(n+1) a_{n+2} (x-1)^n,$$

thus, after plugging these equalities into the initial differential equation, we obtain

$$x \sum_{n=0}^{+\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=0}^{+\infty} (n+1) a_{n+1} (x-1)^n + x \sum_{n=0}^{+\infty} a_n (x-1)^n = 0.$$

In order to have factors $x-1$ instead of x, we add and subtract appropriate expressions:

$$\begin{aligned}(x-1)\sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}(x-1)^n + \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}(x-1)^n \\ + \sum_{n=0}^{+\infty}(n+1)a_{n+1}(x-1)^n + (x-1)\sum_{n=0}^{+\infty}a_n(x-1)^n + \sum_{n=0}^{+\infty}a_n(x-1)^n = 0\end{aligned}$$

$$\begin{aligned}\sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}(x-1)^{n+1} + \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}(x-1)^n \\ + \sum_{n=0}^{+\infty}(n+1)a_{n+1}(x-1)^n + \sum_{n=0}^{+\infty}a_n(x-1)^{n+1} + \sum_{n=0}^{+\infty}a_n(x-1)^n = 0\end{aligned}$$

$$\begin{aligned}\Leftrightarrow \sum_{n=1}^{+\infty}(n+1)na_{n+1}(x-1)^n + \sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}(x-1)^n \\ + \sum_{n=0}^{+\infty}(n+1)a_{n+1}(x-1)^n + \sum_{n=1}^{+\infty}a_{n-1}(x-1)^n + \sum_{n=0}^{+\infty}a_n(x-1)^n = 0\end{aligned}$$

$$\begin{aligned}\Leftrightarrow \sum_{n=1}^{+\infty}(n+1)na_{n+1}(x-1)^n + 2a_2 + \sum_{n=1}^{+\infty}(n+2)(n+1)a_{n+2}(x-1)^n \\ + a_1 + \sum_{n=1}^{+\infty}(n+1)a_{n+1}(x-1)^n + \sum_{n=1}^{+\infty}a_{n-1}(x-1)^n + a_0 + \sum_{n=1}^{+\infty}a_n(x-1)^n = 0\end{aligned}$$

$$\begin{aligned}\Leftrightarrow 2a_2 + a_1 + a_0 \\ + \sum_{n=1}^{+\infty}[(n+1)(n+1)a_{n+1} + (n+2)(n+1)a_{n+2} + a_{n-1} + a_n](x-1)^n = 0\end{aligned}$$

$$\Leftrightarrow \begin{cases} 2a_2 + a_1 + a_0 = 0 \\ (n+2)(n+1)a_{n+2} + a_{n-1} + a_n + (n+1)^2 a_{n+1} = 0. \end{cases}$$

Thus the recurrence relation is

$$\boxed{a_2 = -\frac{a_0 + a_1}{2}, \; a_{n+2} = -\frac{a_{n-1} + a_n + (n+1)^2 a_{n+1}}{(n+1)(n+2)}, \; n \in \mathbb{N}}.$$