M244: Solutions to Final Exam Review

1. Solve (use any method if not otherwise specified):

   (a) $(2x - 3x^2)\frac{dx}{dt} = t\cos(t)$
   SOLUTION: As a separable DE:

$$\int 2x - 3x^2\, dx = \int t\cos(t)\, dt \Rightarrow x^2 - x^3 = \cos(t) + t\sin(t) + C$$

   You can leave your answer in implicit form.

   (b) $y'' + 2y' + y = \sin(3t)$
   SOLUTION: Get the homogeneous part then the particular solution (or use Laplace):

$$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1, -1 \Rightarrow y_h(t) = \mathrm{e}^{-t}\left(C_1 + C_2 t\right)$$

   For the particular solution, (Undet Coefs), $y_p = A\cos(3t) + B\sin(3t)$. Substitute to get:

$$(A - 6B - 9A)\sin(3t) + (B + 6A - 9B)\cos(3t) = \sin(3t)$$

   so that $A = -3/50$, $B = -2/25$. Altogether,

$$y(t) = \mathrm{e}^{-t}\left(C_1 + C_2 t\right) - \frac{3}{50}\cos(3t) - \frac{2}{25}\sin(3t)$$

   (c) $y' = y(y-1)$
   SOLUTION: This is separable. Note that we'll need partial fractions:

$$\int \frac{1}{y(y-1)}\, dy = \int dt \quad \Rightarrow \quad \int -\frac{1}{y} + \frac{1}{y-1}\, dy = t + C \quad \Rightarrow \quad -\ln|y| + \ln|y-1| = t + C$$

   It's good practice to be able to isolate y from this:

$$\ln\frac{|y-1|}{|y|} = t + C \quad \Rightarrow \quad \frac{y-1}{y} = A\mathrm{e}^t \quad \Rightarrow \quad y(1 - A\mathrm{e}^t) = 1 \quad \Rightarrow \quad y(t) = \frac{1}{1 - A\mathrm{e}^t}$$

   (d) $y'' - 3y' + 2y = \mathrm{e}^{2t}$
   SOLUTION: Same technique as 1b. The roots to the characteristic equation are $r = 1, 2$, so the homogeneous part of the solution is:

$$y_h(t) = C_1\mathrm{e}^t + C_2\mathrm{e}^{2t}$$

   Initially, we guess that $y_p(t) = A\mathrm{e}^{2t}$, but that is part of y_h, so multiply by t: $y_p = At\mathrm{e}^{2t}$. Now substitute into the D.E. to get: $A = 1$. The full solution is

$$y(t) = C_1\mathrm{e}^t + C_2\mathrm{e}^{2t} + t\mathrm{e}^{2t}$$

   (e) $y' = \sqrt{t}\mathrm{e}^{-t} - y$.
   SOLUTION: This is a linear differential equation, with integrating factor; $y' + y = \sqrt{t}\mathrm{e}^{-t}$. The integrating factor is $\mathrm{e}^{\int 1\, dt} = \mathrm{e}^t$. Therefore,

$$(y\mathrm{e}^t)' = \sqrt{t} \Rightarrow y\mathrm{e}^t = \frac{2}{3}t^{3/2} + C \Rightarrow y = \left(\frac{2}{3}t^{3/2} + C\right)\mathrm{e}^{-t}$$

   (f) $x' = 2 + 2t^2 + x + t^2 x$.
   SOLUTION: This is a linear differential equation: $x' - (1+t^2)x = 2(1+t^2)$. The integrating factor is:

$$\mathrm{e}^{\int -(1+t^2)\, dt} = \mathrm{e}^{-t-(1/3)t^3}$$

   So we solve the following (to integrate, let $u = t + (1/3)t^2$)

$$\left(x\mathrm{e}^{-t-(1/3)t^3}\right)' = 2(1+t^2)\mathrm{e}^{-t-(1/3)t^3} \Rightarrow \left(x\mathrm{e}^{-t-(1/3)t^3}\right) =$$

$$-2\mathrm{e}^{-t-(1/3)t^3} + C \Rightarrow x = -2 + C\mathrm{e}^{t+(1/3)t^3}$$

(g) $\begin{aligned} x_1' &= 2x_1 + 3x_2 \\ x_2' &= 4x_1 + x_2 \end{aligned}$

SOLUTION: The characteristic equation is $\lambda^2 - 3\lambda - 10 = 0$, which factors as $(\lambda - 5)(\lambda + 2) = 0$. Solving for eigenvectors:

- For $\lambda = -2$, we have $(2 - -2)v_1 + 3v_2 = 0$. Therefore, $\mathbf{v}_1 = (-3, 4)$ (NOTE: Take this as shorthand for a column vector in 2-d)
- For $\lambda = 5$, we have $(2 - 5)v_1 + 3v_2 = 0$, $\mathbf{v}_2 = (1, 1)$.

Overall, the general solution is

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-2t} \begin{bmatrix} -3 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{5t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(h) $(y\cos(x) + 2x\mathrm{e}^y) + (\sin(x) + x^2\mathrm{e}^y - 1)y' = 0$

SOLUTION:

$$M = y\cos(x) + 2x\mathrm{e}^y \quad \Rightarrow \quad M_y = \cos(x) + 2x\mathrm{e}^y \qquad N = \sin(x) + x^2\mathrm{e}^y - 1 \quad \Rightarrow \quad N_x = \cos(x) + 2x\mathrm{e}^y$$

Therefore, this equation is exact. There are multiple ways of determining the solution. For example, we can integrate N with respect to y first:

$$f(x, y) = \int \sin(x) + x^2\mathrm{e}^y - 1\, dy = y\sin(x) + x^2\mathrm{e}^y - y + h(x)$$

To find the arbitrary function of x, differentiate with respect to x to see if we get M:

$$f_x(x, y) = y\cos(x) + 2x\mathrm{e}^y + h'(x)$$

We see that $h'(x) = 0$, so our full solution is:

$$y\sin(x) + x^2\mathrm{e}^y - y = C$$

2. Solve: $t^2y'' - 2ty' - 10y = 0$ using the ansatz $y = t^r$. Substitution gives us:

$$r(r-1) - 2r - 10 = 0 \quad \Rightarrow \quad r = -5, 2$$

That means $y_1 = t^5$ and $y_2 = t^{-2}$ are two solutions to the homogeneous equation. The full solution is then $c_1y_1 + c_2y_2$, or:

$$y(t) = C_1t^5 + C_2t^{-2}$$

3. Show that with $v = y/x$, the following equation becomes separable. NOTE: You do not need to solve the differential equation.

$$\frac{dy}{dx} = \frac{3x - 4y}{y - 2x}$$

SOLUTION: There are several ways to do this. One is to divide numerator and denominator by x, and let $v = \frac{y}{x}$ (the equation is a homogeneous equation), therefore $y = xv$ and $y' = xv' + v$. Although we don't need to solve the DE, we should go ahead and separate the variables.

$$xv' + v = \frac{3 - 4v}{v - 2} \quad \Rightarrow \quad xv' = \frac{3 - 4v - v(v-2)}{v - 2} = \frac{3 - 2v - v^2}{v - 2} \quad \Rightarrow \quad \frac{v - 2}{3 - 2v - v^2}\, dv = \frac{1}{x}\, dx$$

4. Show that with $w = y^3$, the following equation becomes linear in W. NOTE: You do not need to solve the differential equation.

$$\frac{dy}{dx} + 3xy = \frac{x}{y^2}$$

SOLUTION: This is a Bernoulli equation. We could multiply through by y^2 to get y^3 in the middle (not necessary, there are lots of ways to do the substitution)

$$y^2y' + 3xy^3 = x$$

Let $w = y^3$. Then $w' = 3y^2 y'$, or $\frac{1}{3}w' = y^2 y'$. Therefore, we get:

$$\frac{1}{3}w' + 3xw = x \quad \Rightarrow \quad w' + 9xw = 3x$$

which is linear in w.

5. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:

   SOLUTION:

   $$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

   The general solution (before initial conditions):

   $$y(t) = C_1 e^{3t} + C_2 e^{-2t}$$

   With the initial conditions,

   $$1 = C_1 + C_2 \quad \alpha = 3C_1 - 2C_2 \Rightarrow C_1 = \frac{2+\alpha}{5}, \quad C_2 = \frac{3-\alpha}{5}$$

   Therefore,

   $$y(t) = \frac{2+\alpha}{5} e^{3t} + \frac{3-\alpha}{5} e^{-2t}$$

   For $y(t) \to 0$, we must have $\alpha = -2$ (to zero out the first term).

6. If $y' = y(1-y)(2-y)(3-y)(4-y)$ and $y(0) = 2.5$, determine what y does as $t \to \infty$.

   SOLUTION: The idea here is to figure out where the equilibria are and if they are stable or unstable. This is a fifth degree polynomial which increases after $y = 4$ and decreases to $-\infty$ as y decreases. From a quick sketch of the polynomial, we see that $y = 1, 3$ are stable and $y = 0, 2, 4$ are unstable equilibria. If the solution starts at $y = 2.5$, it will increase to the next stable equilibrium $y = 3$.

7. If y_1, y_2 are a fundamental set of solutions to

   $$t^2 y'' - 2y' + (3+t)y = 0$$

   and if $W(y_1, y_2) = 3$, find $W(y_1, y_2)(4)$.

   SOLUTION: Before using Abel's Theorem, put the equation in standard form as:

   $$y'' - \frac{2}{t^2} y' + \frac{3+t}{t^2} y = 0$$

   so that the Wronskian between any two solutions is:

   $$C e^{2 \int t^{-2}\, dt} = C e^{-2/t}$$

   Given that the Wronskian is 3 at $t = 2$, we have:

   $$C e^{-1} = 3 \quad \Rightarrow \quad C = 3e$$

   and now

   $$W(y_1, y_2)(4) = 3e e^{-2/4} = 3\sqrt{e}$$

8. (i) What is the Wronskian? How is it used?

   SOLUTION: The Wronskian is an operation that is performed on two functions f, g:

   $$W(f, g) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$$

   The Wronskian is primarily used to determine if two solutions to a second order linear homogeneous differential equation form a *fundamental set* of solutions- That is, if an arbitrary solution to an IVP could be written as a linear combination of the functions in the set.

(ii) Explain Abel's Theorem:

SOLUTION: Abel's Theorem states that, given any two solutions to a second order linear homogeneous DE (on the interval on which the existence and uniqueness theorem holds):

$$y'' + p(t)y' + q(t)y = 0$$

Then the Wronskian can be computed by:

$$W(y_1, y_2)(t) = C\mathrm{e}^{-\int p(t)\, dt}$$

This implies that either the Wronskian is always zero on the interval, or never zero (the two functions form a fundamental set).

9. Give the two Existence and Uniqueness Theorems we have had in class.

   SOLUTION: I'll go ahead and list all three, although I was just asking for the first two:

   We had the general existence and uniqueness theorem, then one for linear first order, then one for linear second order equations:

   - Let $y' = f(t, y)$ with $y(t_0) = y_0$. If there is a open rectangle R that contains (t_0, y_0), and on which f and $\partial f/\partial y$ are continuous, then there exists an $\epsilon > 0$ for which a unique solution exists to the DE, valid for $t_0 - \epsilon < t < t_0 + \epsilon$.
   - Let $y' + p(t)y = g(t)$, with $y(t_0) = y_0$. If there is an open interval I containing t_0 on which both p, g are continuous, then there is a solution to the IVP, valid on all of I.
   - Let $y'' + p(t)y' + q(t)y = g(t)$, with $y(t_0) = y_0, y'(t_0) = v_0$. If there is an open interval I containing t_0 on which both p, q, g are continuous, then there is a solution to the IVP, valid on all of I.

10. Let $y'' - 6y' + 9y = F(t)$. For each $F(t)$ listed, give the *form* of the general solution using undet. coeffs (do not solve for the coefficients).

    SOLUTION: Before we start, we should go ahead and solve the homogeneous equation:

    $$r^2 - 3r + 9 = 0 \quad \Rightarrow \quad (r-3)^2 = 0 \quad \Rightarrow \quad r = 3, 3$$

    Therefore,

    $$y_h = C_1\mathrm{e}^{3t} + C_2 t\mathrm{e}^{3t}$$

    (a) $F(t) = 2t^2$ SOLUTION: Guess the full quadratic, $y_p = At^2 + Bt + C$

    (b) $F(t) = t\mathrm{e}^{-3t}\sin(2t)$: SOLUTION: We need a polynomial of degree 1 with each sine and cosine:

    $$y_p(t) = \mathrm{e}^{-3t}((At + B)\cos(2t) + (Ct + D)\sin(2t))$$

    (c) $F(t) = t\sin(2t) + \cos(2t)$ SOLUTION: Similar to the last one, but we can take both sine and cosine together:

    $$y_p = (At + B)\cos(2t) + (Ct + D)\sin(2t)$$

    (d) $F(t) = 2t^2 + 12\mathrm{e}^{3t}$ SOLUTION: Break this one up into two solutions- One for $g_1 = t^2$ and one for $g_2 = 12\mathrm{e}^{3t}$. Then:

    $$y_{p_1}(t) = At^2 + Bt + C \qquad y_{p_2} = At^2\mathrm{e}^{3t}$$

    where I multiplied the second guess by t^2 so that it looks like no term of the homogeneous part of the solution.

11. Newton's Law of Cooling states that the rate of change of the temperature of a body is proportional to the difference between the body's temperature and the temperature of the environment.

    In notation, if $u(t)$ is the temperature at time t, and T is the (constant) environmental temperature, then:

    $$\frac{du}{dt} = -k(u - T)$$

    (The negative sign out front implies that k can be assumed to be positive).

    To solve it, we can write

    $$u' = -ku + kT \quad \Rightarrow \quad u(t) = C\mathrm{e}^{-kt} + T$$

12. A spring is stretched 0.1 m by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 0.05 m below its resting equilibrium and released with a downward velocity of 0.1 m/s, determine its position u at time t.

    SOLUTION: From what is given, we can compute the spring constant.

$$mg - kL = 0 \quad \Rightarrow \quad 3 - \frac{k}{10} = 0 \quad \Rightarrow \quad k = 30$$

    Now, we also assume that damping is proportional to the velocity, so:

$$\gamma v = 5\gamma = 3 \quad \Rightarrow \quad \gamma = \frac{3}{5}$$

    Now we can construct the model (downward is positive):

$$2u'' + \frac{3}{5}u + 30u = 0 \qquad u(0) = 0.05 \qquad u'(0) = 0.1$$

    We'll go ahead and use some numerical approximations for this- The roots of the characteristic equation are approximately:

$$r = -0.15 \pm 3.87\, i$$

    (Sorry! This came out of Section 3.7 so I thought the numbers would work out better...) Therefore, the solution is:

$$\mathrm{e}^{-0.15t}(C_1 \cos(3.87\, t) + C_2 \sin(3.87\, t))$$

    If we put in the initial conditions, we get:

$$C_1 = 0.05 \qquad -0.15C_1 + 3.87C_2 = 0.1 \quad \Rightarrow \quad C_2 \approx 0.065$$

    which gives the solution.

13. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$. Find the recurrence relation and write the first 5 terms of the expansion of y.

    SOLUTION: Use a power series based at $x_0 = 1$:

$$y(x) = \sum_{n=0}^{\infty} c_n(x-1)^n \qquad y'(x) = \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2}$$

    Substitute these into the ODE:

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-1)^{n-2} - x\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} - \sum_{n=0}^{\infty} c_n(x-1)^n = 0$$

    For the middle sum, recall our trick: $x = 1 + (x-1)$, which gives us a way of incorporating the x into the sum:

$$x\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} = (1 + (x-1))\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} = \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} + \sum_{n=1}^{\infty} nc_n(x-1)^n$$

    Now shift the index of every sum to match, and you should get the recurrence relation:

$$C_{k+2} = \frac{1}{k+2}\left(C_{k+1} + C_k\right)$$

    We get:

$$y(x) = C_0 + C_1(x-1) + \frac{C_0 + C_1}{2}(x-1)^2 + \frac{C_0 + 3C_1}{6}(x-1)^3 + \frac{2C_0 + 3C_1}{12}(x-1)^4 + \ldots$$

    We can check our answer using the next exercise...

14. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$ (the same as the previous DE), with $y(1) = 1$ and $y'(1) = 2$. Compute the first 5 terms of the power series solution by first computing $y''(1), y'''(1), y^{(4)}(1)$.

First, let's compute the derivatives:

$$y'' = xy' + y \qquad \text{at } x = 1 \qquad \Rightarrow y''(1) = y'(1) + y(1)$$

so that:

$$y''' = xy'' + 2y' \qquad \text{at } x = 1 \qquad \Rightarrow y'''(1) = y''(1) + 2y'(1)$$

and:

$$y^{(4)} = xy''' + 3y'' \qquad \text{at } x = 1 \qquad \Rightarrow y^{(4)}(1) = y'''(1) + 3y''(1)$$

From this, we see that:

$$y''(1) = 3, \qquad y'''(1) = 7, \qquad y^{(4)}(1) = 16$$

Writing out the solution:

$$y(x) = 1 + 2(x-1) + \frac{3}{2}(x-1)^2 + \frac{7}{6}(x-1)^3 + \frac{16}{24}(x-1)^4 + \ldots$$

15. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

$$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6 - t, & 2 < t \end{cases}$$

SOLUTION:

$$\int_0^\infty f(t)\mathrm{e}^{-st}\,dt = 3\int_0^2 \mathrm{e}^{-st}\,dt + \int_2^\infty (6-t)\mathrm{e}^{-st}\,dt =$$

$$\frac{3}{s}(1 - \mathrm{e}^{-2s}) + \frac{\mathrm{e}^{-2s}}{s^2}(4s - 1)$$

16. Determine the Laplace transform:

(a) $t^2\mathrm{e}^{-9t} \Rightarrow \dfrac{2}{(s+9)^3}$

(b) $u_5(t)(t-2)^2 \Rightarrow \quad \mathrm{e}^{-5s}\left(\frac{2}{s^2} + \frac{6}{s^2} + \frac{9}{s}\right)$

(c) $\mathrm{e}^{3t}\sin(4t) \Rightarrow \dfrac{4}{(s-3)^2 + 16}$

(d) $\mathrm{e}^t\delta(t-3) \Rightarrow \mathrm{e}^{-3s+3}$

17. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2 - 4s + 6}$. Rewrite: $2\cdot\dfrac{s-2}{(s-2)^2+2} + \dfrac{3}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{(s-2)^2+2}$ The inverse is then $\mathrm{e}^{2t}\left(2\cos(\sqrt{2}t + \dfrac{3}{\sqrt{2}}\sin(\sqrt{2}t)\right)$

(b) $\dfrac{7}{(s+3)^3} \Rightarrow \dfrac{7}{2}t^2\mathrm{e}^{-3t}$

(c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)}$. You might rewrite this as $\mathrm{e}^{-2s}H(s)$, where

$$H(s) = \frac{4s+2}{(s-1)(s+2)} = \frac{2}{s+2} + \frac{2}{s-1}$$

Now, $h(t) = 2\mathrm{e}^{-2t} + 2\mathrm{e}^t$, and the solution is $u_2(t)h(t-2)$.

(d) $\dfrac{3s-2}{(s-4)^2-3}$ We might rewrite this as:

$$3\cdot\frac{s-4}{(s-4)^2-3}+\frac{10}{\sqrt{3}}\cdot\frac{\sqrt{3}}{(s-4)^2-3}=3F(s-4)+\frac{10}{\sqrt{3}}G(s-4)$$

where $F(s)=\dfrac{s}{s^2-3}, G(s)=\dfrac{\sqrt{3}}{s^2-3}$. The inverse is (Item 14 from the Table):

$$e^{4t}\left(3f(t)+\frac{10}{\sqrt{3}}g(t)\right)=e^{4t}\left(3\cosh(\sqrt{3}t)+\frac{10}{\sqrt{3}}\sinh(\sqrt{3}t)\right)$$

18. Solve the given initial value problems using Laplace transforms:

(a) $y''+2y'+2y=4t$, $y(0)=0$, $y'(0)=-1$. The Laplace transform:

$$Y(s)=\frac{4-s^2}{s^2(s^2+2s+2)}=-\frac{2}{s}+\frac{2}{s^2}+\frac{2s+1}{s^2+2s+2}=$$

$$-\frac{2}{s}+\frac{2}{s^2}+2\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}$$

so that

$$y(t)=-2+2t+e^{-t}\left(2\cos(t)-\sin(t)\right)$$

(b) $y''-2y'-3y=u_1(t)$, $y(0)=0$, $y'(0)=-1$ Use partial fractions:

$$Y(s)=-\frac{1}{4}\cdot\frac{1}{s+1}+\frac{1}{4}\cdot\frac{1}{s-3}+e^{-s}\left(-\frac{1}{3}\cdot\frac{1}{s}+\frac{1}{4}\cdot\frac{1}{s+1}+\frac{1}{12}\cdot\frac{1}{s-3}\right)$$

Think of this second term as $e^{-s}\cdot H(S)$, where

$$h(t)=-\frac{1}{3}+\frac{1}{4}e^{-t}+\frac{1}{12}e^{3t}$$

and the solution is:

$$y(t)=-\frac{1}{4}e^{-t}+\frac{1}{4}e^{3t}+u_1(t)h(t-1)$$

(c) $y''-4y'+4y=t^2e^t$, $y(0)=0$, $y'(0)=0$.

$$Y(s)=\frac{2}{(s-1)^3(s-2)^2}=\frac{2}{(s-1)^3}\cdot\frac{1}{(s-2)^2}=F(s)G(S)$$

Where $f(t)=t^2e^t$ and $g(t)=te^{2t}$. Therefore,

$$y(t)=t^2e^t * te^{2t}$$

19. Consider

$$t^2y''-4ty'+6y=0$$

Using $y_1=t^2$ as one solution, find y_2 by computing the Wronskian two ways.

SOLUTION: We are verifying the solution above by using the Wronskian. Using this technique, we have:

$$\begin{vmatrix} t^2 & y_2 \\ 2t & y_2' \end{vmatrix}=t^2y_2'-2ty_2$$

And, using Abel's Theorem (with $p(t)=-\frac{4}{t}$):

$$W=Ce^{\int\frac{4}{t}\,dt}=Ct^4$$

Therefore,

$$t^2 y_2' - 2ty_2 = Ct^4 \quad \Rightarrow \quad y_2' - \frac{2}{t} y_2 = Ct^2$$

This has an integrating factor of t^{-2}:

$$\left(\frac{y_2}{t^2}\right)' = C \quad \Rightarrow \quad \frac{y_2}{t^2} = Ct + C_1 \quad \Rightarrow \quad y_2 = Ct^3 + C_1 t^2$$

We already have $y_1 = t^2$, so $y_2 = t^3$ (as before).

20. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

(a) $y' - \frac{1}{2}y = e^{2t} \qquad y(0) = 1$. This is a linear (integrating factor) differential equation; the solution will be valid for all time t.

$$y(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{\frac{1}{2}t}$$

(b) $y' = \frac{1}{2}y(3 - y)$.
SOLUTION: Separable. We'll need partial fractions to integrate.

$$\int \frac{1}{3} \cdot \frac{1}{y} + \frac{1}{3} \cdot \frac{1}{3-y}\, dy = \frac{1}{2}t + C \Rightarrow \ln|y| - \ln|3-y| = \frac{3}{2}t + C_2$$

Now solve for y:

$$\frac{y}{3-y} = Ae^{3/2t} \Rightarrow y = \frac{3}{1 + Be^{-3/2t}}$$

If the initial condition is positive, this is valid for all time (Draw the phase diagram and direction field for this autonomous DE to see why). If the initial condition is negative, we would need to find where (in positive time) the solution has a vertical asymptote. NOTE: We are assuming that $t \geq 0$.

(c) $y'' + 2y' + y = 0, \quad y(0) = \alpha, y'(0) = 1$

$$y(t) = \alpha e^{-t} + (1 + \alpha)te^{-t}$$

This solution is valid for all t.

(d) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$ This is an exact equation:

$$\frac{\partial}{\partial y}(2xy^2 + 2y) = 4xy + 2 = \frac{\partial}{\partial x}(2x^2y + 2x)$$

Recall that the solution will be (implicit) $F(x, y) = C$, where

$$F_x = 2xy^2 + 2y \Rightarrow F(x, y) = x^2y^2 + 2xy + h(x)$$

and

$$F_y = 2x^2y + 2x \Rightarrow F(x, y) = x^2y^2 + 2xy + g(y)$$

Comparing, we see $F(x, y) = x^2y^2 + 2xy$, and the implicit solution is:

$$x^2y^2 + 2xy = C$$

Here we will not be able to give an interval on which the solution is valid unless we isolate y, although we would have a requirement that $2x^2y + 2x \neq 0$, so that y' would be defined.

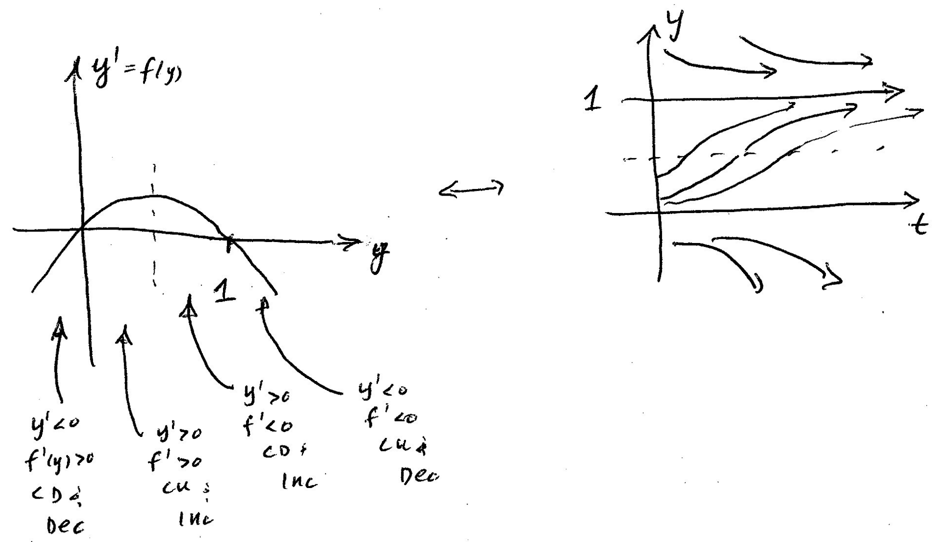
(e) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1$.

$$y(t) = \frac{1}{5}\sin(2t) - \frac{19}{40}\cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

The solution is valid for all t.

21. Suppose $y' = -ky(y-1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

Your graph should be an upside down parabola (vertex up). There are equilibrium solutions at $y = 0$ (unstable) and $y = 1$ (stable).



To find the analytic solution, we might go as follows (there are other ways to integrate this):

$$\int \frac{dy}{y(y-1)} = \int -k\,dt \quad \Rightarrow \quad -\ln|y| + \ln|y-1| = -kt + C \quad \Rightarrow \quad \frac{y}{y-1} = A\mathrm{e}^{-kt}$$

so that

$$y(t) = \frac{1}{1 - A\mathrm{e}^{-kt}}$$

22. True or False (and explain): Every separable equation is also exact. If true, is one way easier to solve over the other?

True:

$$\frac{dy}{dx} = f(y)g(x) \quad \Rightarrow \quad -g(x)\,dx + \frac{1}{f(y)}dy = 0$$

So, letting $-g(x) = M(x,y)$ and $1/f(y) = N(x,y)$, then $M_y = N_x = 0$. Therefore, every separable equation is exact.

To see if either is simpler, check to see what integrals must be performed. Treating the equation as separable, we have to compute the two integrals:

$$\int \frac{dy}{f(y)} \qquad \int g(x)\,dx$$

Treating the equations as exact,

$$\int M(x,y)\,dx = -\int g(x)\,dx + H(y)$$

$$\int N(x,y)\,dy = \int \frac{dy}{f(y)} + G(x)$$

so we have to compute the same integrals either way (so neither is easier than the other).

23. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

This is separable: $y' = y^2(2+x) \Rightarrow y^{-2}\,dy = (2+x)\,dx$, so

$$y(x) = \frac{-2}{x^2+4x-2}$$

This has vertical asymptotes at $x = -2 \pm \sqrt{6}$, so that the solution is valid only when $-2-\sqrt{6} < x < -2+\sqrt{6}$, or when x is approximately between -4.45 and 0.45. Between these vertical asymptotes, y has a minimum where its derivative is 0,

$$y' = y^2(2+x) = 0 \Rightarrow y = 0 \text{ or } x = -2$$

From our solution, we see that $y \neq 0$, so the minimum occurs at $x = -2$, and the minimum is: 1/3.

24. Rewrite the following differential equations as an equivalent system of first order equations. If it is an IVP, also determine initial conditions for the system.

    (a) $y'' - 3y' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$.
    SOLUTION: Let $x_1 = y$ and $x_2 = y'$:

    $$\begin{array}{rl} x_1' &= x_2 \\ x_2' &= -4x_1 + 3x_2 \end{array} \qquad \Rightarrow \qquad \mathbf{x}(0) = \left[\begin{array}{c} 1 \\ 2 \end{array}\right]$$

    (b) $y''' - 2y'' - y' + 4y = 0$
    SOLUTION: Same idea as before, but now we need three variables. Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$. Then:

    $$\begin{array}{rlll} x_1' &= & x_2 & \\ x_2' &= & & x_3 \\ x_3' &= -4x_1 & x_2 & +2x_3 \end{array}$$

    (c) $y'' - yy' + t^2 = 0$
    SOLUTION: This equation is nonlinear and non-autonomous, but the substitutions work just as before: Let $x_1 = y, x_2 = y'$. Then:

    $$\begin{array}{rl} x_1' &= x_2 \\ x_2' &= x_1x_2 - t^2 \end{array}$$

25. Convert one of the variables in the following systems to an equivalent higher order differential equation, and solve it (be sure to solve for both x and y):

    $$\begin{array}{rl} x' &= 4x + y \\ y' &= -2x + y \end{array}$$

    SOLUTION: Solve the first equation for y, substitute into the second equation to get:

    $$x'' - 5x' + 6x = 0 \quad \Rightarrow \quad (r-3)(r-2) = 0$$

    Therefore, $x(t) = C_1\mathrm{e}^{3t} + C_2\mathrm{e}^{2t}$. To find y, we have to compute $x' - 4x$, which is:

    $$y(t) = 3C_1\mathrm{e}^{3t} + 2C_1\mathrm{e}^{2t} - 4(C_1\mathrm{e}^{3t} + C_2\mathrm{e}^{2t})$$

    Therefore, putting these together as a parametric solution:

    $$\left[\begin{array}{c} x \\ y \end{array}\right] = C_1\mathrm{e}^{3t}\left[\begin{array}{c} 1 \\ -1 \end{array}\right] + C_2\mathrm{e}^{2t}\left[\begin{array}{c} 1 \\ -2 \end{array}\right]$$

26. Solve the previous system by using eigenvalues and eigenvectors.

    SOLUTION: As a hint, we should be able to read the eigenvalues and eigenvectors off of the given solution, but we'll go through it the long way just to be double check the method.

    $$A = \left[\begin{array}{cc} 4 & 1 \\ -2 & 1 \end{array}\right] \qquad \begin{array}{rl} \mathrm{Tr}(A) &= 5 \\ \det(A) &= 6 \end{array} \qquad \Rightarrow \qquad \lambda^2 - 5\lambda + 6 = 0$$

And this is indeed the same characteristic function as before. We'll work with $\lambda = 3$ first, and:

$$\begin{array}{rl} (4-3)v_1 + v_2 & = 0 \\ -2v_1 + (1-3)v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} v_1 & = v_1 \\ v_2 & = -v_1 \end{array} \quad \Rightarrow \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

A similar computation shows that $[1, -2]^T$ is an eigenvector for $\lambda = 2$. Therefore, we get the same general solution as before.

27. Verify by direct substitution that the given power series is a solution of the differential equation:

$$y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$(x+1)y'' + y' = 0$$

SOLUTION: If you get stuck on this one, you might think about what this function is without the sum:

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Therefore,

$$y' = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}$$

$$y'' = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots = \sum_{n=2}^{\infty} (-1)^{n+1} (n-1) x^{n-2}$$

So you can check that $(1+x)y'' + y' = 0$ for these terms- However, we should be able to show it in general using the sum notation.

We need to show that the following polynomial is zero for all x:

$$(1+x) \sum_{n=2}^{\infty} (-1)^{n+1} (n-1) x^{n-2} + \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}$$

which is easiest if we have a single sum:

$$\sum_{n=2}^{\infty} (-1)^{n+1} (n-1) x^{n-1} + \sum_{n=2}^{\infty} (-1)^{n+1} (n-1) x^{n-2} + \sum_{n=1}^{\infty} (-1)^{n+1} x^{n-1}$$

Combining to have x^k with $k = 0, 1, 2, \cdots$, we have:

$$\sum_{k=0}^{\infty} \left(k(-1)^{k+2} + (k+1)(-1)^{k+3} + (-1)^{k+2} \right) x^k$$

The coefficients become:

$$(k+1)(-1)^{k+2} + (k+1)(-1)^{k+3}$$

so, if k is even, the expression is 0 and if k is odd the expression is zero- Every coefficient is zero.

28. Convert the given expression into a single power series:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + 2 \sum_{n=2}^{\infty} n a_n x^{n-2} + 3 \sum_{n=1}^{\infty} a_n x^n$$

SOLUTION: The first sum starts with x^2, the second sum starts with x^0 and the third sum starts with x^1.

It will probably be easiest to make the combined sum start with x^1, so we'll do that by first making the individual sums do that:

$$\sum_{n=1}^{\infty} n(n-1)a_n x^n + \left[4a_2 + 2\sum_{n=3}^{\infty} n a_n x^{n-2}\right] + 3\sum_{n=1}^{\infty} a_n x^n$$

If we make m our new index, then in the first and last sums, $n = m$ and in the middle sum, $m = n - 2$:

$$4a_2 + \sum_{m=1}^{\infty} \left[m(m-1)a_m + 2(m+2)a_{m+2} + 3a_m\right] x^m$$

29. Find the recurrence relation for the coefficients of the power series solution to $y'' - (1+x)y = 0$ at $x_0 = 0$.

SOLUTION: Substitute the power series into the DE to get:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - (1+x)\sum_{n=0}^{\infty} a_n x^n = 0$$

Break up the second sum:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

We want to collect these into a single sum- Looks like we might need to peel the constant terms off of the first and second sums to get all series to start with the first power:

$$2 \cdot 1 \cdot a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \left(a_0 + \sum_{n=1}^{\infty} a_n x^n\right) - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now we can write everything as a single sum:

$$(2a_2 - a_0) + \sum_{k=1}^{\infty} \left((k+2)(k+1)a_{k+2} - a_k - a_{k-1}\right) x^k = 0$$

from which we get the recurrence relation:

$$a_2 = \frac{1}{2}a_0 \qquad \text{and } a_{k+2} = \frac{a_k + a_{k-1}}{(k+2)(k+1)} \qquad \text{for } k = 1, 2, 3, \cdots$$

30. Find the first 5 non-zero terms of the series solution to $y'' - (1+x)y = 0$ if $y(0) = 1$ and $y'(0) = -1$ (use derivatives).

SOLUTION: Taking derivatives,

$$y'' = (1+x)y \qquad y^{(3)} = y + (1+x)y' \qquad y^{(4)} = 2y' + (1+x)y''$$

and so on. From this we see that (remember to divide by the appropriate factorial):

$$y(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{40}x^5 + \cdots$$

EXTRA: You might notice that the derivatives have the relation:

$$y^{(k+2)}(0) = ky^{(k-1)}(0) + y^{(k)}(0)$$

Substituting from the Taylor formula $\frac{y^{(k)}}{k!} = a_k$, or $y^{(k)} = k!a_k$ we get:

$$(k+2)!a_{k+2} = k(k-1)!a_{k-1} + k!a_k$$

Simplifying, we get

$$a_{k+2} = \frac{a_k + a_{k-1}}{(k+1)(k+2)}$$

which you might recognize from the previous problem!

31. Let $y'' + \omega^2 y = \cos(\alpha t)$.

   (a) What values of ω, α will result in *beating*? Write the homogenous part of the solution, then give the *form* of the particular part of the solution from Method of Undetermined Coefficients.
   SOLUTION: To get beating, $\omega \approx \alpha$. Note that the circular frequency of a beat is $|\omega - \alpha|$. The general solution to the ODE will be:
   $$y(t) = y_h + y_p$$
   where
   $$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \qquad y_p = A\cos(\alpha t) + B\sin(\alpha t)$$

   (b) Repeat the first part, except for *resonance*.
   SOLUTION: To get resonance, $\omega = \alpha$, and
   $$y_h = C_1 \cos(\omega t) + C_2 \sin(\omega t) \qquad y_p = t(A\cos(\alpha t) + B\sin(\alpha t))$$
   Algebraically, multiplication by t blows up the solution (makes it become unbounded).

32. Let $y'' + \alpha y' + y = 0$. Find (all) values of α for which the solution is *underdamped*, *overdamped*, and *critically damped*.

   SOLUTION: These values depend on the discriminant of the characteristic equation, $r^2 + \alpha r + 1 = 0$. In this case, the discriminant is $\alpha^2 - 4$. We should note that in this context, we're looking at a spring-mass system, so that $\alpha \geq 0$, but that should have been noted in the problem...

   - If $0 \leq \alpha < 2$, then the DE is UNDERDAMPED (complex roots).
   - If $\alpha = 2$, the system is CRITICALLY DAMPED.
   - If $\alpha > 2$, the system is OVERDAMPED.

33. Let $y'' + y' + y = \cos(2t)$.

   (a) If we complexify the problem, how is the right side of the equation changed? How is the ansatz changed?
   SOLUTION: In complexifying the problem, we replace $\cos(2t)$ by e^{2it}, and then assume $y_p = A\mathrm{e}^{2it}$ for some constant A. HOWEVER, since we started with the cosine, we want the real part of $A\mathrm{e}^{2it}$ (if we started with $\sin(2t)$, then we would want the imaginary part of y_p).

   (b) Using your previous answer, find the amplitude and the phase shift of the forced response, $y_p(t) = R\cos(\omega t - \delta)$.
   SOLUTION: We showed in class that, if
   $$A = \frac{1}{\alpha + \beta i} \quad \Rightarrow \quad R = \frac{1}{|\alpha + \beta i|} \qquad \delta = \tan^{-1}(\beta/\alpha)$$
   So we need the constant A. To do that, substitute $y = A\mathrm{e}^{2it}$ into the DE and solve for the constant A:
   $$A\mathrm{e}^{2it}(-4 + 2i + 1) = \mathrm{e}^{2it} \quad \Rightarrow \quad A = \frac{1}{-3 + 2i}$$
   Therefore, the amplitude R and phase angle δ are given by
   $$R = \frac{1}{\sqrt{9+4}} = \frac{1}{\sqrt{13}} \qquad \delta = \tan^{-1}\left(-\frac{2}{3}\right) + \pi$$
   where we add π because the point $(-3, 2)$ is in Quadrant II.

34. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$
   $$y(t) = \frac{-y_0}{y_0 t^2 - 1}$$
   If $y_0 > 0$, then the solution will only be valid between $\pm\frac{1}{\sqrt{y_0}}$. If $y_0 < 0$, the solution will be valid for all t.

35. Solve the linear system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, if A is as defined below:

(a) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$ SOLN: $\lambda = \pm 2i$

For $\lambda = 2i$, if we use the second equation (not necessary, but perhaps better scaled), then $-v_1 + (-2-2i)v_2) = 0$, so $\mathbf{v} = \begin{bmatrix} -2-2i \\ 1 \end{bmatrix}$.

Now we compute $e^{\lambda t}\mathbf{v}$ before writing the solution:

$$(\cos(2t) + i\sin(2t)) \begin{bmatrix} -2-2i \\ 1 \end{bmatrix} = \begin{bmatrix} (-2\cos(2t) + 2\sin(2t)) + i(-2\cos(2t) - 2\sin(2t)) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

The solution to the DE is now:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} -2\cos(2t) + 2\sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} -2\cos(2t) - 2\sin(2t) \\ \sin(2t) \end{bmatrix}$$

(b) $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ SOLN: $\lambda^2 - 3\lambda - 4 = 0$, so $\lambda = -1, 4$.

If $\lambda = -1$, then $3v_1 + 3v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. If $\lambda = 4$, then $-2v_1 + 3v_2 = 0$, or $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Therefore, the general solution is:

$$\mathbf{x}(t) = C_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$

You should see that $\lambda = -3, -3$. In this case, we don't need the general eigenvector (we'll find a specific eigenvector below). If its not specified, we defined $\mathbf{v}_0$ as (x_0, y_0), and we write the particular part in terms of this initial condition by getting $\mathbf{w}$:

$$\begin{array}{ll} (a-\lambda)x_0 + by_0 & = w_1 \\ cx_0 + (d-\lambda)y_0 & = w_2 \end{array} \quad \Rightarrow \quad \begin{array}{ll} 6x_0 + 18y_0 & = w_1 \\ 2x_0 - 6y_0 & = w_2 \end{array}$$

The general solution is then:

$$\mathbf{x}(t) = e^{-3t} \left[\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 6x_0 + 18y_0 \\ 2x_0 - 6y_0 \end{bmatrix} \right]$$

36. For each system $\mathbf{x}' = A\mathbf{x}$, the matrix A depends on the parameter α. Find how the classification of the origin changes depending on α.

(a) $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$ $\quad \begin{array}{c} \mathrm{Tr}(A) = 0 \\ \det(A) = -4 + 5\alpha \\ \Delta = -4(-4 + 5\alpha) \end{array}$

SOLUTION: On the Poincare Diagram, we're located on the determinant axis. Therefore, we have a center if the determinant is positive ($\alpha > 4/5$), we have "uniform motion" if $\alpha = \frac{4}{5}$, and we have a saddle if $\alpha < 4/5$.

(b) $\begin{bmatrix} \alpha & 2 \\ 3 & 1 \end{bmatrix}$ $\quad \begin{array}{c} \mathrm{Tr}(A) = \alpha + 1 \\ \det(A) = \alpha - 6 \\ \Delta = \alpha^2 - 2\alpha + 25 \end{array}$

SOLUTION: We would probably need to do a sign analysis like we did in class, but if we look at the discriminant, we see that it is always positive (it has complex roots). Therefore, we stay outside of the parabola in the Poincare Diagram.

We might check the other signs- The trace changes sign at $\alpha = -1$ and the determinant changes sign at $\alpha = 6$. Putting these on a number line:

Tr(A)	$--$	$++$	$++$
det(A)	$--$	$--$	$++$
	$\alpha < -1$	$-1 < \alpha < 6$	$\alpha > 6$

Now we can summarize:

$\alpha < -1$	Saddle (Quadrant III)
$\alpha = -1$	Saddle
$-1 < \alpha < 6$	Saddle (Quadrant IV)
$\alpha = 6$	Line of unstable fixed points
$\alpha > 6$	Source (Quadrant I)