Review Questions, Exam 1, Math 244

These questions are presented to give you an idea of the variety and style of question that will be on the exam. It is not meant to be exhaustive, so be sure that you understand the homework problems and quizzes.

Discussion Questions:

1. True or False, and explain:
   (a) If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.
   (b) If $ty' + 2y = \cos(t)$, then the existence and uniqueness theorem will say that we have a unique solution to this DE with any initial value.
   (c) Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them (you may assume df/dy is continuous).
   (d) If $y' = \cos(y)$, then the solutions are periodic.
   (e) All autonomous equations are separable.
   (f) All separable equations are exact.

Solve:

Give the general solution if there is no initial value. Before giving the solution, state what kind of differential equation it is (linear, separable, exact)- Multiple classes are possible, just give the one you will use.

1. $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{x}$
2. $(x + y)\,dx - (x - y)\,dy = 0$. Use the substitution $v = y/x$ (homogeneous)
3. $\dfrac{dy}{dx} = \dfrac{2x + y}{3 + 3y^2 - x} \quad y(0) = 0.$
4. $\dfrac{dy}{dx} = -\dfrac{2xy + y^2 + 1}{x^2 + 2xy}$
5. $\frac{dy}{dt} = 2\cos(3t) \qquad y(0) = 2$
6. $y' - \frac{1}{2}y = 0 \qquad y(0) = 200$. Additionally, give the interval on which the solution is valid.
7. $y' = (1 - 2x)y^2 \qquad y(0) = -1/6$. Additionally, give the interval on which the solution is valid.
8. $y' - \frac{1}{2}y = e^{2t} \qquad y(0) = 1$
9. $y' = \frac{1}{2}y(3 - y)$
10. $\sin(2t)\,dt + \cos(3y)\,dy = 0$
11. $y' = xy^2$
12. $\dfrac{dy}{dx} - \dfrac{3}{2x}y = \dfrac{2x}{y}$. Use the substitution $v = y^2$ (Bernoulli).
13. $y' + 2y = g(t)$, $y(0) = 0$ and
$$g(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$$

Misc.

1. Show that $u(x,t) = \mathrm{e}^{-\alpha^2 t}\sin(x)$ is a solution to the PDE: $u_t = \alpha^2\, u_{xx}$.

2. Determine all values of r for which $y = t^r$ is a solution to the DE: $t^2 y'' - 4ty' + 4y = 0$.

3. If $y(t) = C\mathrm{e}^{2t} + t^2$ is the solution to $y' = 2y + g(t)$, what must $g(t)$ be?

4. The population of mice ($P(t)$) in a field after t years satisfies the following IVP:

$$\frac{dP}{dt} = 3P\left(1 - \frac{P}{2500}\right) \qquad P(0) = 1000$$

   (a) What is the population when it is growing the fastest?

   (b) What is the carrying capacity?

   (c) Without solving the DE, what happens to our population as $t \to \infty$?

   (d) What is the significance of the "3" in the equation?

5. We want to construct a new population model. In this model, there is a "minimum population" of 1 (actually, 100, but we'll scale it) so that, if the population ever goes below this number, then there is not enough population to recover, and the population goes extinct. There is also a larger number that we called the environmental threshold, of 3. Note that there should also be an equilibrium population of 0.

   We want to find the simplest model that will have the desired behavior. First, noting that it is autonomous, graph $y' = f(y)$, then write down a possible polynomial for f.

6. Suppose we have a tank that contains 100 L of water, in which there is 5 kg of salt. Brine is pouring into the tank at a concentration of 2 kg per liter, and at a rate of 4 liters per minute. The well mixed solution leaves the tank at a rate of 4 liters per minute.

   Write the initial value problem that describes the amount of salt in the tank at time t, and solve.

7. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank?

8. Modify the tank problem so that the well mixed solution leaves the tank at a rate of 6 liters per minute. In this case, just write down the IVP, you don't need to solve it.

9. Going back again to the original tank problem- If we change the rate at which the the well mixed solution leaves the tank to 2 liters per minute, what is the new IVP? (You don't need to solve the IVP).

10. Newton's Law of Cooling states that the rate of change of the temperature of a body is proportional to the difference between the temperature of the body and the environment (which we assume is some constant). Write down a differential equation which represents this statement, then find the general solution.

11. Assume our ambient temperature is 20 degrees Celsius. My cup of coffee is initially at 50 degrees. Suppose we also know that after 2 minutes, the coffee is at 40 degrees. What equation would we need to solve in order to determine the minutes it takes for the coffee to become 30 degrees? (If we were allowed calculators, I'd ask you to solve it, but since we're not, just set up the equation).

12. Suppose you borrow \$10000.00 at an annual interest rate of 5%. If you assume continuous compounding and continuous payments at a rate of k dollars per month, set up a model for how much you owe at time t in years. Give an equation you would need to solve if you wanted to pay off the loan in 10 years.

13. Show that the IVP $xy' = y - 1$, $y(0) = 2$ has no solution. (Note: Part of the question is to think about how to show that the IVP has no solution).

14. Suppose that a certain population grows at a rate proportional to the square root of the population. Assume that the population is initially 400 (which is 20^2), and that one year later, the population is 625 (which is 25^2). Determine the time in which the population reaches 10000 (which is 100^2).

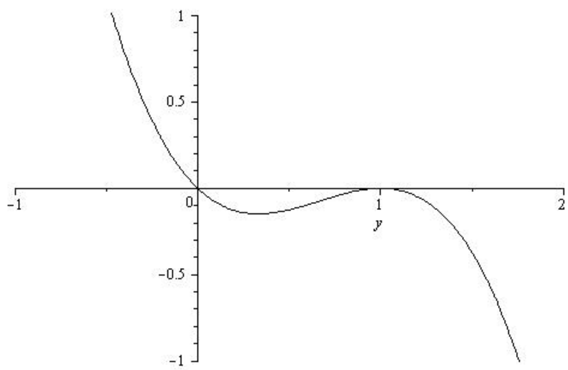
15. Consider the sketch below of $F(y)$, and the differential equation $y' = F(y)$.

    (a) Find and classify the equilibrium.

    (b) Find intervals (in y) on which $y(t)$ is concave up.

    (c) Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down.

    (d) Find an appropriate polynomial for $F(y)$.

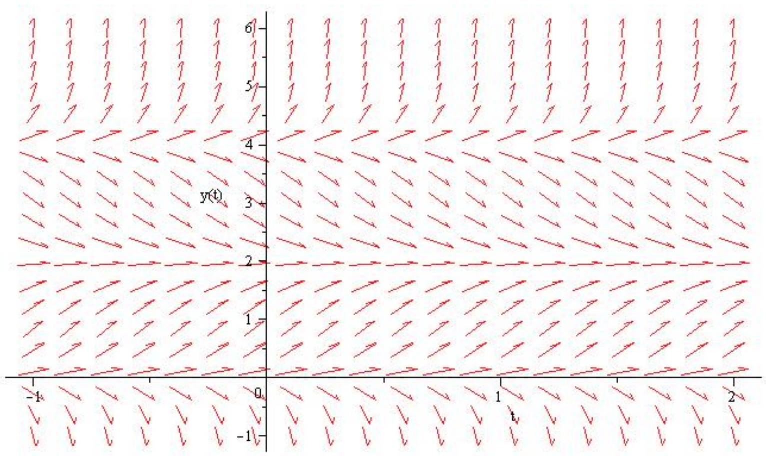


16. Consider the IVP: $t(t-5)y' + \cos(t)y = t^2$, with $y(1) = 2$. Using the existence and uniqueness theorem, does there exist a unique solution to the IVP? If so, what is the full interval on which it exists?

17. Given the DE below, state where in the $ty-$plane the hypotheses of the existence and uniqueness theorem are satisfied:

$$y' = \frac{1 + t^2}{3y - y^2}$$

18. Given the direction field below, find a differential equation that is consistent with it.



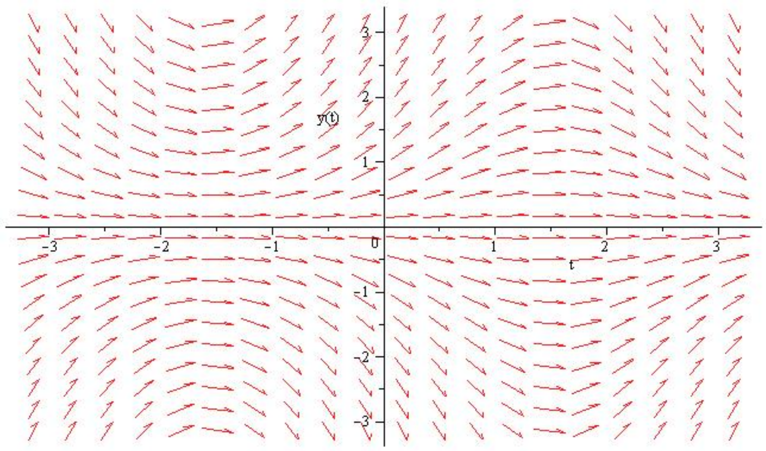
19. Consider the direction field below, and answer the following questions:

(a) Is the DE possibly of the form $y' = f(t)$?

(b) Is the DE possible of the form $y' = f(y)$?

(c) Is there an equilibrium solution? (If so, state it):

(d) Draw the solution corresponding to $y(-1) = 1$.



20. Evaluate the following integrals:

$$\int x^3 e^{2x}\,dx \qquad \int \frac{x}{(x-1)(2-x)}\,dx \qquad e^{-\int 3/t\,dt}$$