Complex to Real Solutions

Given
$$ay'' + by' + cy = 0$$
and assuming that the solutions to the characteristic equation are complex conjugates,
$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \lambda \pm \mu i$$
so that
$$\lambda = -\frac{b}{2a} \qquad \mu = \frac{\sqrt{4ac - b^2}}{2a}$$
we show that the following two functions form a fundamental set of solutions to $ay'' + by' + cy = 0$:
$$y_1 = \mathrm{e}^{\lambda t}\cos(\mu t) \qquad y_2 = \mathrm{e}^{\lambda t}\sin(\mu t)$$
First, we show that they are each solutions, so differentiate and substitute back into the DE. Note that every term will have the exponential attached, so we could factor it out:
$$y_1' = \mathrm{e}^{\lambda t}\left(\lambda\cos(\mu t) - \mu\sin(\mu t)\right) \qquad y_2' = \mathrm{e}^{\lambda t}\left(\mu\cos(\mu t) + \lambda\sin(\mu t)\right)$$
$$y_1'' = \mathrm{e}^{\lambda t}\left((\lambda^2 - \mu^2)\cos(\mu t) - 2\lambda\mu\sin(\mu t)\right) \qquad y_2'' = \mathrm{e}^{\lambda t}\left(2\lambda\mu\cos(\mu t) + (\lambda^2 - \mu^2)\sin(\mu t)\right)$$
Substitute for y_1, and factor out the coefficients of $\cos(\mu t)$ and $\sin(\mu t)$. We show that each of these are zero. For $\cos(\mu t)$, we have:
$$a(\lambda^2 - \mu^2) + b\lambda + c = a\lambda^2 + b\lambda + c - a\mu^2$$
Substitute in for $\lambda = -b/2a$ and μ:
$$= a\frac{b^2}{4a^2} - \frac{b^2}{2a} + c - a\frac{4ac - b^2}{4a^2} = \frac{ab^2}{4a^2} - \frac{2ab^2}{4a^2} + \frac{4a^2c}{4a^2} - \frac{4a^2c - ab^2}{4a^2} = 0$$
Similarly, one can do the algebra to show that y_2 is also a solution.
Now that we have two solutions, we show that they are linearly independent for any interval (the constants a, b, c are continuous everywhere):
$$W(y_1, y_2)(t) = \begin{vmatrix} \mathrm{e}^{\lambda t}\cos(\mu t) & \mathrm{e}^{\lambda t}\sin(\mu t) \\ \mathrm{e}^{\lambda t}\left(\lambda\cos(\mu t) - \mu\sin(\mu t)\right) & \mathrm{e}^{\lambda t}\left(\mu\cos(\mu t) + \lambda\sin(\mu t)\right) \end{vmatrix} =$$
$$\mathrm{e}^{2\lambda t}\left(\mu\cos^2(\mu t) + \lambda\cos(\mu t)\sin(\mu t) - \lambda\cos(\mu t)\sin(\mu t) + \mu\sin^2(\mu t)\right) = \mu\mathrm{e}^{2\lambda t}$$
This is not zero as long as μ is not zero. Since we assumed that we had complex roots, $\mu \neq 0$.