General Review Questions

NOTE: You should also review past quizzes and exams and be sure that you can solve systems using eigenvalues/eigenvectors (Homework sets 1 and 2), and analyze nonlinear systems (Set 3 and the last quiz).

For the exam, you will be given the table of Laplace Transforms, the Poincaré Diagram, and I will write the formulas for the Variation of Parameters formula on the board (the differential equations that u_1, u_2 must satisfy).

You may use a calculator, but not one that can perform symbolic computations (like factoring, differentiating, etc.). The length of the exam will be about an exam and a half.

1. Solve (use any method if not otherwise specified):
   (a) $(2x - 3x^2)\frac{dx}{dt} = t\cos(t)$
   (b) $y'' + 2y' + y = \sin(3t)$
   (c) $y'' - 3y' + 2y = \mathrm{e}^{2t}$
   (d) $x' = \sqrt{t}\mathrm{e}^{-t} - x$
   (e) $x' = 2 + 2t^2 + x + t^2 x$

2. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:
$$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

3. The Wronskian of two functions is $W(t) = t^2 - 4$. Are they two linearly independent solutions to a second order linear differential equation? Why or why not?

4. Compute $\mathcal{L}(\cos(t))$ by using the definition of the Laplace transform.

5. Show that $y_1(t) = t$, $y_2(t) = t^2$ are linearly independent using the *definition* of linear independence. Compute the Wronskian of y_1 and y_2: Can they be linearly independent *solutions* to a second order linear differential equation?

6. Let $y''' - y' = t\mathrm{e}^{-t} + 2\cos(t)$. First, use our ansatz to find the characteristic equation for the third order homogeneous equation. Determine a suitable form for the particular solution, y_p using Undetermined Coefficients. Do not solve for the coeffs.

7. A tank contains 200 gallons of water with 100 pounds of salt. Water containing 1 pound of salt per gallon is entering at a rate of 3 gallons per minute. The well-mixed solution is pumped out at a rate of 2 gallons per minute. Find the concentration of salt in the tank at time t (assuming the tank can hold it).

8. Suppose that we have a mass-spring system modelled by the differential equation
$$x'' + 2x' + x = 0, x(0) = 2, x'(0) = -3$$
Find the solution, and determine whether the mass ever crosses $x = 0$. If it does, determine the velocity at that instant. See if it crosses if the initial velocity is cut in half.

9. Let $y(x)$ be a power series solution to $(1-x)y'' + y = 0,\ x_0 = 0$. Find the recurrence relation, and write the first 5 terms of the expansion of y.

10. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$. Find the recurrence relation and write the first 5 terms of the expansion of y.

11. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$ (the same as the previous DE), with $y(1) = 1$ and $y'(1) = 2$. Compute the first 5 terms of the power series solution by first computing $y''(1), y'''(1), y^{(4)}(1)$.

12. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:
$$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6 - t, & 2 < t \end{cases}$$

13. Determine the Laplace transform:
    (a) $t^2\mathrm{e}^{-9t}$
    (b) $\mathrm{e}^{2t} - t^3 - \sin(5t)$
    (c) $u_5(t)(t-5)^4$
    (d) $\mathrm{e}^{3t}\sin(4t)$
    (e) $\mathrm{e}^{t}\delta(t-3)$
    (f) $t^2 u_4(t)$

14. Find the inverse Laplace transform:
    (a) $\dfrac{2s - 1}{s^2 - 4s + 6}$
    (b) $\dfrac{7}{(s+3)^3}$
    (c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)}$
    (d) $\dfrac{3s - 2}{(s-4)^2 - 3}$

15. Solve the given initial value problems using Laplace transforms:

    (a) $y'' + 2y' + 2y = 4t$, $y(0) = 0$, $y'(0) = -1$
    (b) $y'' + 9y = 10e^{2t}$, $y(0) = -1$, $y'(0) = 5$
    (c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$
    (d) $y'' - 4y' + 4y = t^2 e^t$, $y(0) = 0$, $y'(0) = 0$ (You may write the solution as a convolution)

16. Evaluate: $\displaystyle\int_0^\infty \sin(3t)\delta(t - \frac{\pi}{2})\, dt$

17. If $y'(t) = \delta(t - c)$, what is $y(t)$?

18. What was the *ansatz* we used to obtain the characteristic equation for $ay'' + by' + cy = 0$? for $\mathbf{x}' = A\mathbf{x}$?

19. Given that $y_1(t) = t^2$ is one solution to the differential equation $t^2y'' - 4ty' + 6y = 0$, find a second linearly independent solution, y_2 (HINT: The Wronskian can be computed two ways).

20. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

    (a) $y' - 0.5y = e^{2t} \qquad y(0) = 1$
    (b) $y'' + 4y' + 5y = 0, \quad y(0) = 1, y'(0) = 0$
    (c) $y' = 1 + y^2$
    (d) $y' = \frac{1}{2}y(3 - y)$
    (e) $\sin(2x)dx + \cos(3y)dy = 0$
    (f) $y'' + 2y' + y = 2e^{-t}, \quad y(0) = 0, y'(0) = 1$
    (g) $y' = xy^2$
    (h) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$
    (i) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1.$

21. Suppose $y' = -ky(y - 1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

22. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

23. If $y(t)$ is a population at time t, what is the model for "exponential growth"? What is the model for growth with a "carrying capacity" in the environment or equivalently, what is the logistic model?

24. We have two tanks, A and B with 20 and 30 gallons of fluid, respectively. Water is being pumped into Tank A at a rate of 2 gallons per minute, 2 ounces of salt per gallon. The well-mixed solution is pumped out of Tank A and into Tank B at a rate of 4 gallons per minute. Solution from Tank B is entering Tank A at a rate of 2 gallons per minute. Water is being pumped into Tank B at k gallons per minute with 3 ounces of salt per gallon. The solution is being pumped out of tank B at a total rate of 5 gallons per minute (2 of them are going into tank A).

    - What should k be in order for the amount of solution in Tank B to remain at 30? Use this value for the remaining problems.
    - Write the system of differential equations for the amount of salt in Tanks A, B at time t. Do not solve.
    - Find the equilibrium solution and classify it.

25. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

26. For each nonlinear system, find and classify the equilibria:

    (a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 + 2y \\ 1 - 3x^2 \end{bmatrix}$
    (b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 - y \\ x^2 - y^2 \end{bmatrix}$
    (c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x + x^2 + y^2 \\ y(1 - x) \end{bmatrix}$
    (d) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x - y^2 \\ y - x^2 \end{bmatrix}$

27. Be sure to know the Existence and Uniqueness Theorem for $y' = f(t, y)$ and $y'' + p(t)y' + q(t)y = f(t)$.

    For example, if $y' = y^{1/3}$, $y(0) = 0$, find two solutions to the IVP. Why does this not violate the existence and uniqueness theorem?