

Review Solutions

1. State the existence and uniqueness theorem.

SOLUTION:

Let $y'' + p(t)y' + q(t)y = g(t)$, with $y(t_0) = y_0$ and $y'(t_0) = v_0$. Then if p, q and g are all continuous on an open interval I containing t_0 , a unique solution exists to the IVP, valid for all t in I .

2. Finish the definition: Functions f, g are linearly independent if:
the only solution to $k_1f(t) + k_2g(t) = 0$ is $k_1 = k_2 = 0$ on an interval I .
3. If the $W(y_1, y_2) = t^2$, can y_1, y_2 be two independent solutions to $y'' + p(t)y' + q(t)y = 0$? Explain.

They can, if the solution is only valid for $t > 0$ or $t < 0$.

This is due to Abel's Theorem (actually, a corollary on Pg. 156), which said that, if two functions are solutions to a linear second order equation, then the Wronskian is either always zero or never zero *on the interval for which the solutions are valid*.

4. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

The operator is found by getting all terms in y to one side of the equation, everything else on the other. In this case, we have:

$$L(y) = y' - y^2$$

This is not a linear operator. We can check using the definition:

$$L(cy) = cy' - c^2y^2 \neq cL(y)$$

Furthermore,

$$L(y_1 + y_2) = (y_1' + y_2') - (y_1 + y_2)^2 \neq L(y_1) + L(y_2)$$

5. Find the solution to the initial value problem:

$$u'' + u = \begin{cases} 3t & \text{if } 0 \leq t \leq \pi \\ 3(2\pi - t) & \text{if } \pi < t < 2\pi \\ 0 & \text{if } t \geq 2\pi \end{cases} \quad u(0) = 0 \quad u'(0) = 0$$

Without regards to the initial conditions, we can solve the three nonhomogeneous equations. In each case, the homogeneous part of the solution is $c_1 \cos(t) + c_2 \sin(t)$.

- $u'' + u = 3t$. We would start with $y_p = At + B$. Substituting, we get: $At + B = 3$, so $y_p = 3t$. The general solution in this case is:

$$u(t) = c_1 \cos(t) + c_2 \sin(t) + 3t$$

- $u'' + u = 6\pi - 3t$. From our previous analysis, the solution is:

$$u(t) = c_1 \cos(t) + c_2 \sin(t) + 6\pi - 3t$$

- The last part is just the homogeneous equation.

The only thing left is to find c_1, c_2 in each of the three cases so that the overall function u is continuous:

- $u(0) = 0, u'(0) = 0 \Rightarrow$

$$u(t) = -3 \sin(t) + 3t \quad 0 \leq t \leq \pi$$

- $u(\pi) = 3$ and $u'(\pi) = 6$, so:

$$u(t) = 9 \sin(t) + (3 - 6\pi) \cos(t) + 6\pi - 3t \quad \pi < t < 2\pi$$

- $u(2\pi) = 3 - 6\pi, u'(2\pi) = 6$:

$$u(t) = 6 \sin(t) + (3 - 6\pi) \cos(t) \quad t \geq 2\pi$$

6. Solve:

$$u'' + \omega_0^2 u = F_0 \cos(\omega t), \quad u(0) = 0 \quad u'(0) = 0$$

if $\omega \neq \omega_0$. (Hint: Probably easiest to use the Method of Undetermined Coefficients)

The homogeneous part of the solution is $c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$. The guess for the particular part is $y_p = A \cos(\omega t) + B \sin(\omega t)$. Substitute y_p into the differential equation and solve:

$$\begin{array}{rcl} \omega_0^2(y_p) & = & A \cos(\omega t) + B \sin(\omega t) \\ y_p'' & = & -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t) \\ \hline F_0 \cos(\omega t) & = & A(\omega_0^2 - \omega^2) \cos(\omega t) + B(\omega_0^2 - \omega^2) \sin(\omega t) \end{array}$$

Therefore, $A = \frac{F_0}{\omega_0^2 - \omega^2}$, and $B = 0$.

The solution is now:

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{\omega_0^2 - \omega^2} \cos(\omega t)$$

Putting in the initial conditions:

$$u(0) = 0 \Rightarrow 0 = c_1 + \frac{F_0}{\omega_0^2 - \omega^2} \Rightarrow c_1 = -\frac{F_0}{\omega_0^2 - \omega^2}$$

And

$$u'(0) = 0 \Rightarrow c_2\omega_0 = 0 \Rightarrow c_2 = 0$$

The solution is:

$$u(t) = \frac{F_0}{\omega_0^2 - \omega^2} (\cos(\omega t) - \cos(\omega_0 t))$$

7. In class, we said that given:

$$u'' + \omega_0^2 u = F_0 \cos(\omega t) \quad u(0) = 0 \quad u'(0) = 0$$

If $\omega \neq \omega_0$, then

$$u(t) = \frac{F_0}{\omega_0^2 - \omega^2} (\cos(\omega t) - \cos(\omega_0 t))$$

Show the solution if $\omega = \omega_0$ two ways:

- Start over, with Method of Undetermined Coefficients

With undetermined coefficients, we take:

$$y_p = (A \cos(\omega_0 t) + B \sin(\omega_0 t)) t$$

We multiply by t since the original guess would have been the solution to the homogeneous equation. Take the first and second derivatives (Hint: Keep track of the sine and cosine coefficients):

$$\begin{aligned} y_p &= At \cos(\omega_0 t) + Bt \sin(\omega_0 t) \\ y_p' &= (A + B\omega_0 t) \cos(\omega_0 t) + (B - A\omega_0 t) \sin(\omega_0 t) \\ y_p'' &= (2B\omega_0 - A\omega_0^2 t) \cos(\omega_0 t) + (-2A\omega_0 - B\omega_0^2 t) \sin(\omega_0 t) \end{aligned}$$

Taking $y_p'' + \omega_0^2 y_p$, we get:

$$F_0 \cos(\omega_0 t) = 2B\omega_0 \cos(\omega_0 t) - 2A\omega_0 \sin(\omega_0 t)$$

so that $A = 0$, $B = \frac{F_0}{2\omega_0}$. Putting the solution together and solving for the coefficients:

$$u(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t) + \frac{F}{2\omega_0} t \sin(\omega_0 t) \quad u(0) = 0 \quad u'(0) = 0$$

we get that $A = 0$ and $B = 0$. Our final answer:

$$u(t) = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$

- Take the limit of the above expression as $\omega \rightarrow \omega_0$.

We can find the function directly by taking the limit (Use L'Hospital's rule, differentiating with respect to ω):

$$\lim_{\omega \rightarrow \omega_0} \frac{F_0(\cos(\omega t) - \cos(\omega_0 t))}{\omega_0^2 - \omega^2} = \lim_{\omega \rightarrow \omega_0} \frac{F_0 \cdot t \sin(\omega_0 t)}{2\omega} = \frac{F_0}{2\omega_0} t \sin(\omega_0 t)$$

8. For the following question, recall that the acceleration due to gravity is 32 ft/sec².

An 8 pound weight is attached to a spring from the ceiling. When the weight comes to rest at equilibrium, the spring has been stretched 2 feet. The damping constant for the system is 1-lb-sec/ft. If the weight is raised 6 inches above equilibrium and given an upward velocity of 1 ft/sec, find the equation of motion for the weight. Write the solution as $R \cos(\omega t - \delta)$, if possible.

SOLUTION: First the constants. Since $mg - kL = 0$, we find that $mg = 8$, so $2k = 8$, or $k = 4$.

We are given that $\gamma = 1$, and since $8 = mg$, then $m = 8/32 = 1/4$.

$$\frac{1}{4}y'' + y' + 4y = 0 \quad y(0) = -\frac{1}{2} \quad y'(0) = -1$$

Or we could write: $y'' + 4y' + 16y = 0$. Solving the characteristic equation, we get

$$r^2 + 4r + 16 = 0 \quad \Rightarrow \quad r = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 16}}{2} = \frac{-4 \pm 4\sqrt{3}i}{2} = -2 \pm 2\sqrt{3}i$$

Therefore, the general solution is:

$$y(t) = e^{-2t} (C_1 \cos(2\sqrt{3}t) + C_2 \sin(2\sqrt{3}t))$$

Solving the IVP, differentiate to get the equations for C_1, C_2 :

$$y' = -2e^{-2t} (C_1 \cos(2\sqrt{3}t) + C_2 \sin(2\sqrt{3}t)) + 2\sqrt{3}e^{-2t} (C_2 \cos(2\sqrt{3}t) - C_1 \sin(2\sqrt{3}t))$$

Therefore,

$$\begin{aligned} -\frac{1}{2} &= C_1 \\ -1 &= -2C_1 + 2\sqrt{3}C_2 \end{aligned} \quad \Rightarrow \quad C_1 = -\frac{1}{2} \quad C_2 = -\frac{1}{\sqrt{3}}$$

Rewrite the function using the formulas given:

$$y(t) = \sqrt{\frac{7}{12}} e^{-2t} \cos(2\sqrt{3}t - \delta)$$

where $(-1/2, -1/\sqrt{3})$ is in quadrant 3, so that

$$\delta = \tan^{-1} \left(\frac{2}{\sqrt{3}} \right) + \pi \approx 3.9987 \text{ rad}$$

9. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

Find a second linearly independent solution, y_2 .

First, rewrite the differential equation in standard form:

$$y'' - \frac{2}{t^2}y = 0$$

Then $p(t) = 0$ and $W(y_1, y_2) = Ce^0 = C$. On the other hand, the Wronskian is:

$$W(y_1, y_2) = \frac{1}{t}y_2' - \frac{1}{t^2}y_2$$

Put these together:

$$\frac{1}{t}y_2' - \frac{1}{t^2}y_2 = C \quad y_2' + \frac{1}{t}y_2 = Ct$$

The integrating factor is t ,

$$(ty_2)' = Ct^2 \quad \Rightarrow \quad ty_2 = C_1 t^3 + C_2 \quad \Rightarrow \quad C_1 t^2 + \frac{C_2}{t}$$

Notice that we have *both* parts of the homogeneous solution, $y_1 = \frac{1}{t}$ and $y_2 = t^2$.

10. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped*? *underdamped*?

The model equation can be written as:

$$0.01u'' + 0.05u' + ku = 0 \quad \Rightarrow \quad u'' + 5u' + \alpha u = 0$$

where $100k = \alpha$. The solutions depend on the discriminant,

$$25 - 4\alpha$$

If this is zero, we have a system that is critically damped. In this case, $k = 4/2500$

If the discriminant is negative, the system is underdamped. Solving for k , we get that $k > 4/2500$.

11. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

(a) $y'' + 4y' + 4y = t^{-2}e^{-2t}$

Using the Variation of Parameters, $y_p = u_1y_1 + u_2y_2$, we have:

$$y_1 = e^{-2t} \quad y_2 = te^{-2t} \quad g(t) = \frac{e^{-2t}}{t^2}$$

with a Wronskian of e^{-4t} . You should find that:

$$u_1' = -\frac{1}{t} \quad u_2' = \frac{1}{t^2}$$

$$u_1 = -\ln(t) \quad u_2 = -\frac{1}{t}$$

so $y_p = -\ln(t)e^{-2t} - e^{-2t}$. This last term is part of the homogeneous solution, so this simplifies to $-\ln(t)e^{-2t}$. Now that we have all the parts,

$$y(t) = e^{-2t}(C_1 + C_2t) - \ln(t)e^{-2t}$$

(b) $y'' - 2y' + y = te^t + 4$, $y(0) = 1$, $y'(0) = 1$.

With the Method of Undetermined Coefficients, we first get the homogeneous part of the solution,

$$y_h(t) = e^t(C_1 + C_2t)$$

Now we construct our ansatz (Multiplied by t after comparing to y_h):

$$g_1 = te^t \Rightarrow y_{p_1} = (At + B)e^t \cdot t^2$$

Substitute this into the differential equation to solve for A, B :

$$y_{p_1} = (At^3 + Bt^2)e^t \quad y_{p_1}' = (At^3 + (3A + B)t^2 + 2Bt)e^t$$

$$y_{p_1}'' = (At^3 + (6A + B)t^2 + (6A + 4B)t + 2B)e^t$$

Forming $y_{p_1}'' - 2y_{p_1}' + y_{p_1} = te^t$, we should see that $A = \frac{1}{6}$ and $B = 0$, so that $y_{p_1} = \frac{1}{6}t^3e^t$.

The next one is a lot easier! $y_{p_2} = A$, so $A = 4$, and:

$$y(t) = e^t(C_1 + C_2t) + \frac{1}{6}t^3e^t + 4$$

with $y(0) = 1$, $C_1 = -3$. Solving for C_2 by differentiating should give $C_2 = 4$. The full solution:

$$y(t) = e^t \left(\frac{1}{6}t^3 + 4t - 3 \right) + 4$$

(c) $y'' + 4y = 3 \sin(2t)$, $y(0) = 2$, $y'(0) = -1$.

The homogeneous solution is $C_1 \cos(2t) + C_2 \sin(2t)$. Just for fun, you could try Variation of Parameters. We'll outline the Method of Undetermined Coefficients:

$$y_p = (A \sin(2t) + B \cos(2t))t = At \sin(2t) + Bt \cos(2t)$$

$$y_p'' = (-4At - 4B) \sin(2t) + (4A - 4Bt) \cos(2t)$$

taking $y_p'' + 4y_p = 3 \sin(2t)$, we see that $A = 0$, $B = -\frac{3}{4}$, so the solution is:

$$y = c_1 \cos(2t) + c_2 \sin(2t) - \frac{3}{4}t \cos(2t)$$

With $y(0) = 2$, $c_1 = 2$. Differentiating to solve for c_2 , we find that $c_2 = -1/8$.

(d) $y'' + 9y = \sum_{m=1}^N b_m \cos(m\pi t)$

The homogeneous part of the solution is $C_1 \cos(3t) + C_2 \sin(3t)$. We see that $3 \neq m\pi$ for $m = 1, 2, 3, \dots$

The forcing function is a sum of N functions, the m^{th} function is:

$$g_m(t) = b_m \cos(m\pi t) \quad \Rightarrow \quad y_{p_m} = A \cos(m\pi t) + B \sin(m\pi t)$$

Differentiating,

$$y_{p_m}'' = -m^2 \pi^2 A \cos(m\pi t) - m^2 \pi^2 B \sin(m\pi t)$$

so that $y_{p_m}'' + 9y_{p_m} = (9 - m^2 \pi^2)A \cos(m\pi t) + (9 - m^2 \pi^2)B \sin(m\pi t)$.

Solving for the coefficients, we see that $A = b_m/(9 - m^2 \pi^2)$ and $B = 0$. Therefore, the full solution is:

$$y(t) = C_1 \cos(3t) + C_2 \sin(3t) + \sum_{m=1}^N \frac{b_m}{9 - m^2 \pi^2} \cos(m\pi t)$$

12. Rewrite the expression in the form $a + ib$:

- $2^{i-1} = e^{\ln(2^{i-1})} = e^{(i-1)\ln(2)} = e^{-\ln(2)} e^{i\ln(2)} = \frac{1}{2} (\cos(\ln(2)) + i \sin(\ln(2)))$
- $e^{(3-2i)t} = e^{3t} e^{-2ti} = e^{3t} (\cos(-2t) + i \sin(-2t)) = e^{3t} (\cos(2t) - i \sin(2t))$
(Recall that cosine is an even function, sine is an odd function).
- $e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1$

13. Find a linear second order differential equation with constant coefficients if

$$y_1 = 1 \quad y_2 = e^{-t}$$

form a fundamental set, and $y_p(t) = \frac{1}{2}t^2 - t$ is the particular solution.

The roots to the characteristic equation are $r = 0$ and $r = -1$. The characteristic equation must be $r(r + 1) = 0$ (or a constant multiple of that). Therefore, the differential equation is:

$$y'' + y' = 0$$

For $y_p = \frac{1}{2}t^2 - t$ to be the particular solution,

$$y_p'' + y_p' = (1) + (t - 1) = t$$

so the full differential equation must be:

$$y'' + y' = t$$

14. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t - 4)y'' + 3ty' + 4y = 2 \quad y(3) = 0 \quad y'(3) = -1$$

Write in standard form:

$$y'' + \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)}$$

The coefficient functions are all continuous on each of three intervals:

$$(-\infty, 0), (0, 4) \text{ and } (4, \infty)$$

Since the initial time is 3, we choose the middle interval, $(0, 4)$.

15. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c .

If $L(3e^{2t}) = -9e^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + e^{2t}$$

We see that: $L(3e^{2t}) = -9e^{2t}$. By linearity,

$$cL(3e^{2t}) = L(3ce^{2t}) = -9ce^{2t} = e^{2t}$$

so c must be $-1/9$, and

$$L(-\frac{1}{9}3e^{2t}) = L(-\frac{1}{3}e^{2t}) = e^{2t}$$

Similarly,

$$L(t^2 + 3t) = 5t^2 + 3t - 16$$

We need to multiply the right-side of the equation by -2 to get the desired part of our solution, so multiply both sides by -2 :

$$-2L(t^2 + 3t) = -10t^2 - 6t + 32$$

By linearity, $-2L(t^2 + 3t) = L(-2t^2 - 6t)$. The particular solution is therefore,

$$y_p(t) = -2t^2 - 6t - \frac{1}{3}e^{2t}$$

16. Show that, using the substitution $x = \ln(t)$, then the differential equation:

$$4t^2 y'' + y = 0$$

becomes a differential equation with constant coefficients.

Solve it.

The way the ODE is written now, the derivative is with respect to t . We need to convert it to a derivative in x :

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \frac{dy}{dx} \cdot \frac{1}{t}$$

And the second derivative:

$$\frac{d^2 y}{dt^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{1}{t} + \frac{dy}{dx} \left(-\frac{1}{t^2} \right) = \frac{d^2 y}{dx^2} \cdot \frac{dx}{dt} \cdot \frac{1}{t} - \frac{dy}{dx} \cdot \frac{1}{t^2}$$

Now defining $y' = \frac{dy}{dx}$, the differential equation becomes:

$$4t^2 \left(y'' \cdot \frac{1}{t^2} - y' \frac{1}{t^2} \right) + y = 4y'' - 4y' + y = 0$$

The characteristic equation has solns: $r = \frac{1}{2}, \frac{1}{2}$

$$y(x) = e^{(1/2)x} (C_1 + C_2 x)$$

Back substituting $x = \ln(t)$, we get:

$$y(t) = \sqrt{t} (C_1 + C_2 \ln(t))$$

17. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \rightarrow \infty$.

The solution is:

$$y = \left(\frac{2 + \alpha}{5} \right) e^{3t} + \left(\frac{3 - \alpha}{5} \right) e^{-2t}$$

For the solution to tend to zero, the first constant must be zero, so $\alpha = -2$.

18. Without using the Wronskian, determine whether $f(x) = xe^{x+1}$ and $g(x) = (4x - 5)e^x$ are linearly independent.

Form the equation that we are solving, and factor/divide out the e^x term:

$$C_1 x e + C_2 4x - 5C_2 = 0$$

This implies that: $-5C_2 = 0$, so C_2 must be zero. The second requirement would be that $eC_1 + 4C_2 = 0$, but with $C_2 = 0$, then C_1 must be zero.

The only solution is $C_1 = C_2 = 0$, so the functions are linearly independent.

19. Given $y'' + p(t)y' + q(t)y = 0$, is it always possible to construct a fundamental set of solutions? (Be specific as to how to do it. You might find the Existence and Uniqueness Theorem useful).

If p, q are continuous on an interval I containing t_0 , then y_1 is constructed as the (unique) solution to:

$$y'' + p(t)y' + q(t)y = 0 \quad y(t_0) = 1 \quad y'(t_0) = 0$$

Similarly, y_2 is constructed as the (unique) solution to:

$$y'' + p(t)y' + q(t)y = 0 \quad y(t_0) = 0 \quad y'(t_0) = 1$$

The initial conditions will force the Wronskian of y_1, y_2 to be nonzero, which gives us a fundamental set of solutions.

(Recall that this is more of a theoretical result rather than a theorem that we actually use to construct the homogeneous solutions).