Complex Numbers

SOLUTIONS to the exercises:

1 Exercises

1. Suppose the roots to a cubic polynomial are $a = 3$, $b = 1 - 2i$ and $c = 1 + 2i$. Compute $(x - a)(x - b)(x - c)$.

   SOLUTION: You could multiply it all out at once. If you take $(x - b)(x - c)$ first, you get:
   $$(x - a)(x - b)(x - c) = (x - 3)(x^2 - 2x + 5) = x^3 - 5x^2 + 11x - 15$$

2. Find the roots to $x^2 - 2x + 10$. Write them in polar form.

   SOLUTION:
   $$x = \frac{2 \pm \sqrt{2^2 - 4(10)}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

3. Show that:
   $$\text{Re}(z) = \frac{z + \bar{z}}{2} \qquad \text{Im}(z) = \frac{z - \bar{z}}{2i}$$

   SOLUTION: If $z = a + bi$, then $\bar{z} = a - bi$, so:
   $$z + \bar{z} = a + ib + (a - ib) = 2a = 2\text{Re}(z)$$
   And similarly,
   $$z - \bar{z} = a + ib - (a - ib) = 2ib \quad \Rightarrow \quad \text{Im}(z) = b = \frac{z - \bar{z}}{2i}$$

4. For the following, let $z_1 = -3 + 2i$, $z_2 = -4i$

   (a) Compute $z_1\bar{z}_2$, z_2/z_1

   SOLUTION:
   $$z_1\bar{z}_2 = (-3 + 2i)(4i) = (8i^2) - 12i = -8 - 12i$$
   $$\frac{z_2}{z_1} = \frac{-4i}{-3 + 2i} \cdot \frac{-3 - 2i}{-3 - 2i} = \frac{-8 + 12i}{3^2 + 4^2} = -\frac{8}{13} + \frac{12}{13}i$$

   (b) Write z_1 and z_2 in polar form.

   SOLUTION: For z_1, the point $(-3, 2)$ is in Quadrant II. To find the argument (angle) for z_1, we have to add π to the arctangent:
   $$\theta = \tan^{-1}\left(\frac{2}{-3}\right) + \pi \approx 2.554 \text{ rad}$$
   The length is: $r = \sqrt{3^2 + 2^2} = \sqrt{13}$.

We can verify this, by checking that:

$$\sqrt{13}\cos(\theta) \approx -3 \qquad \sqrt{13}\sin(\theta) \approx 2$$

Writing the answer in polar form, $re^{i\theta}$,

$$-3 + 2i = \sqrt{13}e^{2.554i}$$

For z_2, it is much simpler: $r = 4$ and $\theta = -\pi/2$. Therefore,

$$-4i = re^{i\theta} = 4e^{-i\pi/2}$$

5. In each problem, rewrite each of the following in the form $a + bi$:

(a) e^{1+2i}

$$e\left(\cos(2) + i\sin(2)\right)$$

(b) e^{2-3i}

$$e^2\left(\cos(-3) + i\sin(-3)\right) = e^2\left(\cos(3) - i\sin(3)\right)$$

(c) $e^{i\pi}$

$$\cos(\pi) + i\sin(\pi) = -1$$

(d) 2^{1-i} First, notice that $2^{1-i} = 2 \cdot 2^{-i}$, so we'll compute 2^{-i} below:

$$2^{-i} = e^{\ln(2^{-i})} = e^{-i\ln(2)} = \cos(-\ln(2)) + i\sin(-\ln(2)) = \cos(\ln(2)) - i\sin(\ln(2))$$

Therefore, 2^{1-i} can be expressed as:

$$2(\cos(\ln(2)) - i\sin(\ln(2)))$$

(e) $e^{2-\frac{\pi}{2}i}$

$$e^2\left(\cos(-\pi/2) + i\sin(-\pi/2)\right) = -ie^2$$

(f) π^i

$$\pi^i = e^{\ln(\pi^i)} = e^{i\ln(\pi)} = \cos(\ln(\pi)) + i\sin(\ln(\pi))$$