Section 2.3- Modeling

1. Problem 3:

   Our first model will be for the amount of salt in the tank at time t, where $0 \le t \le 10$ (since the dynamics change at time $t = 10$). So our initial model is the usual "Rate in-Rate out", where $Q(t)$ is the amount of salt in the tank at t seconds:

   $$\frac{dQ}{dt} = \frac{1}{2} \cdot \frac{\text{lbs}}{\text{gal}} \cdot 2 \cdot \frac{\text{gal}}{\text{min}} - 2\frac{\text{gal}}{\text{min}} \cdot \frac{Q(t)\text{ lbs}}{100\text{ gal}}$$

   Simplifying, the IVP, valid for $0 \le t < 10$ is given by the following (at time zero, there was fresh water in the tank):

   $$\frac{dQ}{dt} = 1 - \frac{1}{50}Q \qquad Q(0) = 0$$

   We give the solution here as if it were linear (it is also separable):

   $$Q' + \frac{1}{50}Q = 1 \Rightarrow \text{ int. factor: } \quad e^{(1/50)t} \quad \Rightarrow \left(Qe^{(1/50)t}\right) = 50e^{(1/50)t} + C$$

   so that

   $$Q(t) = 50 + Ce^{-(1/50)t} \quad \text{IC } Q(0) = 0 \quad \Rightarrow \quad Q(t) = 50 - 50e^{-(1/50)t}$$

   To solve the problem in the exercise, we need to model the next 10 seconds.

   At time $t = 10$, the dynamics change (the rate in is now zero), so the differential equation becomes:

   $$Q' = -\frac{1}{50}Q \qquad \text{valid for } t \ge 10$$

   To get an IVP that we can solve, we need an initial condition: It comes from finding $Q(10)$ from our first model:

   $$Q(10) = 50 - 50e^{-1/5} \approx 9.06343$$

   For $t \ge 10$, the IVP is:

   $$Q' = -\frac{1}{50}Q, \qquad Q(10) \approx 9.06343$$

   Solve the IVP with $Q(10) = 50 - 50e^{-1/5}$:

   $$Q(t) = Ae^{(-1/50)t} \qquad 50 - 50e^{-1/5} = Ae^{-1/5} \qquad A = 50e^{1/5} - 50 \approx 11.07$$

   The solution, for $t \ge 10$, is then:

   $$Q(t) = 11.07e^{-(1/50)t}$$

   And now substitute $t = 20$ to find the final amount of salt in the tank: $Q(20) \approx 7.421$ pounds.

2. Problem 5, part (a) was assigned:

   As usual, let $Q(t)$ be the amount of salt (in ounces) at time t (measured in minutes). Then dQ/dt will be measured in ounces per minute.

   The rate in:

$$\frac{1}{4}\left(1+\frac{1}{2}\sin(t)\right)\cdot\frac{\text{oz}}{\text{gal}}\cdot 2\cdot\frac{\text{gal}}{\text{min}}=\frac{1}{2}\left(1+\frac{1}{2}\sin(t)\right)\frac{\text{oz}}{\text{min}}$$

   The rate out:

$$2\frac{\text{gal}}{\text{min}}\cdot\frac{Q(t)\text{ lbs}}{100\text{ gal}}=\frac{1}{50}Q\text{ lbs/min}$$

   The model equation:

$$\frac{dQ}{dt}=\frac{1}{2}\left(1+\frac{1}{2}\sin(t)\right)-\frac{1}{50}Q \qquad Q(0)=50$$

   The initial condition was that we started with 50 ozs of salt in a tank of 100 gallons.

   This is a linear differential equation, and the integrating factor is the same as computed earlier in Problem 3:

$$\mathrm{e}^{(1/50)t}\left(Q'+\frac{1}{50}Q\right)=\frac{1}{2}\mathrm{e}^{(1/50)t}+\frac{1}{4}\mathrm{e}^{(1/50)t}\sin(t)$$

   Note that we need to *integrate by parts twice*, so that:

$$\int \mathrm{e}^{(1/50)t}\sin(t)\,dt=-\frac{2500}{2501}\mathrm{e}^{(-1/50)t}\cos(t)-\frac{50}{2501}\mathrm{e}^{(-1/50)t}\sin(t)$$

   Now we write the full solution:

$$Q(t)=25-\frac{625}{2501}\cos(t)+\frac{25}{5002}\sin(t)+\frac{63150}{2501}\mathrm{e}^{-(1/50)t}$$

   SORRY about those constants! You could have left them as numerical approximations.

3. Problem 9:

   In the absence of payments, the rate of change of our loan will increase proportionally to the current amount. That is, let $S(t)$ be the amount of money (measured in dollars) owed at time t (measured in years). With no payments,

$$\frac{dS}{dt}=rS$$

   where r is the annual interest rate (annual because we are measuring t in years). By making "continuous payments", at a constant annual rate k:

$$\frac{dS}{dt}=rS-k$$

We can solve this generally, with $S(0) = S_0$:

$$S(t) = \frac{k}{r} + \left(S_0 - \frac{k}{r}\right) e^{rt}$$

Putting in the values, $S(0) = 8000$, $r = 1/10$ and leaving k as an adjustable parameter,

$$S(t) = 10k - (8000 - 10k)\, e^{(1/10)t}$$

We want to find the value of k so that our loan is paid off in three years, or $S(3) = 0$:

$$0 = 10k - (8000 - 10k)e^{3/10} \qquad k \approx 3086.64$$

(Side remark: That's about \$8.42 per day) So over the three year period, we would pay:

$$3 \cdot 3086.64 = 9259.92$$

so the interest paid was about \$1259.92.

4. Problem 12: We are given that:

$$Q' = -rQ \quad \Rightarrow \quad Q(t) = Q_0 e^{-rt}$$

And we are told that the half-life of Carbon-14 is 5730 years. That means that, if Q_0 is the initial amount, then:

$$\frac{1}{2} Q_0 = Q_0 e^{-r \cdot (5730)}$$

Divide both sides by Q_0, and solve for r:

$$r = \frac{\ln(1/2)}{-5730} = \frac{\ln(2)}{5730} \approx 0.00012097 = 1.2097 \times 10^{-4}$$

The general solution is:

$$Q(t) = Q_0 e^{(-1.2907 \times 10^{-4})t}$$

We now think of Q_0 as some unknown (but fixed) amount, and let T be the time it takes to decrease Q_0 to 20% of the original amount. Then solve for T:

$$\frac{1}{5} Q_0 = Q_0 e^{(-1.2907 \times 10^{-4})T}$$

This gives $T \approx 13,304.65$ years.

5. Problem 13:

Let us parse out the problem:

- The population of mosquitoes increases at a rate proportional to the current population...

  If $P(t)$ is the population at time t, so far this says

$$\frac{dP}{dt} = kP \qquad \Rightarrow \qquad P(t) = P_0 e^{kt}$$

- ... and in the absence of other factors, the population doubles each week. If we measure t in days, this means that:

$$2P_0 = P_0 e^{7k} \qquad \Rightarrow \qquad k = \frac{\ln(2)}{7} \approx 0.9902 \text{ per day}$$

  Now, going back to our model: So far, without predation, the rate of change population at time t (in days) is:

$$\frac{dP}{dt} = \frac{\ln(2)}{7} P$$

- There are 200,000 mosquitoes initially (Modeled as $P(0) = 200,000$), and predators eat 20,000 per day- This is a constant decrease:

$$\frac{dP}{dt} = \frac{\ln(2)}{7} P - 20,000, \qquad P(0) = 200,000$$

  Solve using an integrating factor of e^{-kt}:

$$(Pe^{-kt})' = 20000e^{-kt} \qquad \Rightarrow \qquad P(t) = -\frac{20000}{k}\frac{20,000 \cdot 7}{\ln(2)} + \left(200,000 - \frac{20,000 \cdot 7}{\ln(2)}\right) e^{kt}$$

$$P(t) = \frac{20,000 \cdot 7}{\ln(2)} + \left(200,000 - \frac{20,000 \cdot 7}{\ln(2)}\right) e^{\frac{\ln(2)}{7} t}$$

  Side Remark:

$$e^{\frac{\ln(2)}{7} t} = \left(e^{\ln(2)}\right)^{\frac{1}{7} t} = 2^{t/7}$$

- If t is measured in weeks, then things simplify a bit. In that case, $k = \ln(2)$ and:

$$\frac{dP}{dt} = \ln(2)P - 140,000 \qquad P(0) = 200,000$$

  and the solution to the IVP is:

$$P(t) = \frac{140,000}{\ln(2)} + \left(200,000 - \frac{140,000}{\ln(2)}\right) e^{\ln(2)t}$$

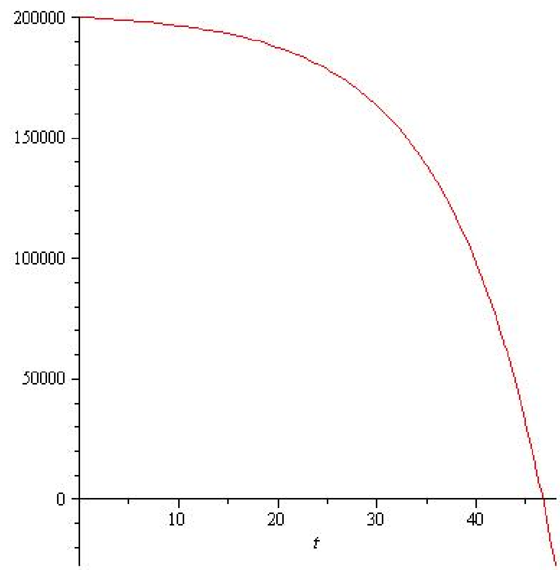

Figure 1: Figure for Exercise 13. We see that the population decreases to zero after approximately 45 weeks or so.

or equivalently

$$P(t) = \frac{140,000}{\ln(2)} + \left(200,000 - \frac{140,000}{\ln(2)}\right) 2^t$$

In numerical form,

$$P(t) = 201,977.31 - 19,77.31 \cdot 2^{-t}$$

Solving for when $P(t) = 0$, we see that the solution is valid for $0 \leq t \leq 6.6745$ (weeks).

6. Problem 23:

   NOTE: We cannot solve $s(t) = 0$ below exactly (using algebra), so we may stop there. The solution below uses computer software to finish the problem (So if you were able to find $s(t)$, that is as far as you needed to go here).

   A Physics Note: If something is measured in pounds, it has the same units as mass times gravity, mg. Gravity in this problem will be measured as 32 feet per second squared. Given that the weight is 180 pounds, and gravity is 32, we can then compute the mass: $180/32 = 5.625$.

   Going back to our model from Section 1.1:

$$ma = m\frac{dv}{dt} = mg - kv \qquad \Rightarrow \qquad \frac{dv}{dt} = g - \frac{k}{m}v$$

Before the parachute opens, $0 \le t \le 10$, we have:

$$\frac{dv}{dt} = -\frac{\frac{3}{4}}{\frac{180}{32}} v + 32 = -\frac{2}{15} v + 32, \qquad v(0) = 0$$

Solving for the velocity equation,

$$v(t) = 240 - 240\mathrm{e}^{-(2/15)t}$$

The speed when the parachute opens ($t = 10$) is $v(10) \approx 176.74$ feet per second.

We can now integrate velocity to find position, $s(t)$. Careful here! A quick analysis of our velocity equation says that the velocity *towards the ground* is positive. But, if we say that $s(0) = 5000$, our height will be increasing (since $v = s' > 0$). To compensate, we set $s(0) = -5000$:

$$s(t) = -6800 + 240t + 1800\mathrm{e}^{-(2/15)t}$$

So the position at $t = 10$ is $s(10) \approx -3925.53$, which we interpret as 3925.53 feet above the ground, so the skydiver has fallen (approximately) $5000 - 3925.53 = 1074.47$ feet

To answer the last two questions, we reformulate the velocity equation. To simplify things, we'll reset the clock to $t = 0$ (interpret as minutes past 10):

$$\frac{dv}{dt} = -\frac{12}{\frac{180}{32}} v + 32 = -\frac{32}{15} v + 32$$

The "equilibrium" is $v(t) = 15$ feet per second. Solve this equation to with an "initial velocity" of 176.74, and:

$$v(t) = 15 + 161.74\mathrm{e}^{-(32/15)t}$$

Integrate to find position, with "initial position" at -3925.53:

$$s(t) = 15t - 75.82\mathrm{e}^{-(32/15)t} - 3849.71$$

To solve for t so that $s = 0$, we will need Maple. Here are the commands to first get an estimate, then solve:

```
S:=15*t-75.82*exp(-(32/15)*t)-3849.71;
plot(S,t=0..260);
fsolve(S=0,t=250..260);
```

and we get that $t \approx 256.6473333$ seconds, or about 4.3 minutes.

7. Problem 28:

   NOTE: In Problem 28, we can find $v(t)$ and $s(t)$ (the velocity and position functions) without the aid of a computer, but beyond that, we would need computer software. Below is the full solution, but we only need to go as far as finding $s(t)$

From what is given in the problem, we'll use our standard model:

$$\frac{dv}{dt} = g - \frac{k}{m}v$$

with $g = 9.8$ meters per seconds squared, $k = 0.2 = \frac{1}{5}$, and $m = 0.25 = \frac{1}{4}$. Therefore,

$$\frac{dv}{dt} = 9.8 - \frac{4}{5}v$$

and, with the initial condition $v(0) = 0$, we have:

$$v(t) = 12.25 - 12.25\mathrm{e}^{-(4/5)t}$$

We can now get position at time t, with the initial position -30:

$$s(t) = -45.31 + 12.25t + 15.31\mathrm{e}^{-(4/5)t}$$

We can now answer the first question, with a little Maple. To find the velocity when the ball hits the ground, we need to find the time at which this happens. Set $s(t) = 0$ and solve for t. The Maple commands are:

```
S:=-45.31+12.25*t+15.31*exp(-(4/5)*t);
plot(S,t=0..5);
fsolve(S=0,t=3..4);
```

From this, $t \approx 3.63$ seconds. Substitute this into velocity:

$$v(3.63) \approx 11.58$$

For part (b), we want the velocity to be no more than 10 meters per second (what is the maximum height from which the ball can be dropped)? First, look at the velocity:

$$v(t) = 12.25 - 12.25\mathrm{e}^{-(4/5)t}$$

This is an increasing function (look at the plot in Maple, or consider that the derivative is positive). Therefore, we will find the time it takes for the velocity to reach 10 meters per second.

$$v(t) = 10 \Rightarrow 12.25 - 12.25\mathrm{e}^{-(4/5)t} = 10$$

Solve for t and get about $t = 2.1182$.

Now, look at the height function, $s(t)$, where $s(0) = S_0$:

$$s(t) = 12.25t + 15.31\mathrm{e}^{-(4/5)t} + (S_0 - 15.31)$$

We want to find S_0 so that $s(2.1182) = 0$. Substitute this value of t in and solve for S_0. You should find that $S_0 \approx -13.45$.

For part (c), we will need to use Maple, but let's see how far we can go before we need it: First we'll need velocity and position in terms of k:

$$\frac{dv}{dt} = 9.8 - (4k)v \qquad v(0) = 0$$

so that the solution in terms of k is:

$$v(t) = \frac{9.8}{4k} - \frac{9.8}{4k}\mathrm{e}^{-4kt}$$

The position at time t (with $s(0) = -30$) is:

$$s(t) = \frac{9.8}{4k}t + \frac{9.8}{16k^2}\mathrm{e}^{-4kt} - \left(30 + \frac{9.8}{16k^2}\right)$$

We now need to solve for k so that, when the ball hits the ground, the velocity is no more than 10. Let $t*$ be the time when the ball hits the ground- It too depends on k. Therefore, we have two equations in two unknowns (the unknowns are k and t^*):

$$v(t^*) = 10 \quad \Rightarrow \quad \frac{9.8}{4k} - \frac{9.8}{4k}\mathrm{e}^{-4kt^*} = 10$$

and

$$s(t^*) = 0 \quad \Rightarrow \quad \frac{9.8}{4k}t^* + \frac{9.8}{16k^2}\mathrm{e}^{-4kt^*} - \left(30 + \frac{9.8}{16k^2}\right) = 0$$

To solve this system of equations in Maple, we'll first define them, then plot the curves (to solve these numerically, we'll need to give Maple an approximate solution). Once we see the point of intersection, then Maple will solve it:

```
Eqn1:=(9.8/(4*k))-(9.8/(4*k))*exp(-4*k*t)=10;
Eqn2:=(9.8/(4*k))*t+(9.8/(16*k^2))*exp(-4*k*t)-(30+(9.8/(16*k^2)))=0;
with(plots):
implicitplot({Eqn1,Eqn2},k=0.1..1,t=0.1..5);
fsolve({Eqn1,Eqn2},{k,t},{k=0.2..0.25,t=3..5});
```

Maple gives the solution as:

```
{k = .2394381624, t = 3.952304030}
```

So we conclude that, if $k \geq 0.2394$, then the ball will hit the ground with a velocity of at most 10 meters per second.