Solutions to Sample Questions, Exam 1

Math 244 Fall 2009

1 Short Answer/True or False

1. True or False, and explain: If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.

   False: (a) Equilibrium solutions are only defined for *autonomous* differential equations, (b) This is an isocline for a slope of zero, and (c) $y = -2t$ is not a solution.

2. State the Existence and Uniqueness Theorem for linear first order initial value problems (IVPs).

   Let $y' + p(t)y = g(t)$ with $y(t_0) = y_0$.

   If p, g are continuous on an open interval I containing t_0, then a unique solution exists to the IVP. In addition, the solution is valid on I.

   (Note: The interval I is a single (connected) interval, not two or more intervals).

3. State the general Existence and Uniqueness Theorem for first order initial value problems (IVPs).

   Let $y' = f(t, y)$ with $y(t_0) = y_0$.

   If f is continuous on an open rectangle containing (t_0, y_0), then a solution exists.

   If $\partial f/\partial y$ is continuous on an open rectangle containing (t_0, y_0), then the solution is unique.

   In the general case, we cannot predict ahead of time on what interval the solution will be valid- We have to solve the IVP.

4. Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them.

   It is, but only if f is not continuous. If f is continuous (which is a normal assumption on f), then it is not possible (draw a picture in the phase plane and you'll see why).

5. What is a *linear* first order differential equation.

   A linear first order DE is any DE that can be expressed as:

   $$y' + p(t)y = g(t)$$

6. Let $y' = \sin(y)$. It is possible that the solution can oscillate (or be periodic).

   False. In the direction field, the slopes cannot depend on t- That is, all of the slopes at a given y value are all the same. This means that solutions are monotonically increasing, monotonically decreasing, or are constant (equilibria). No solution can oscillate.

7. What is an n^{th} order differential equation?

   The order of a differential equation refers to the integer of the highest derivative. An n^{th} order DE would have an n^{th} derivative as the highest derivative.

8. What's the difference between:

   "The domain of $y(t)$" and "The time interval for which $y(t)$ is a solution to the DE"?

   When we talk about the time interval on which a solution is valid, the interval must be a single (connected) interval. A domain can be any combination of points, intervals, etc.

9. Let $\frac{dy}{dt} = 1 + y^2$. Then the solution will be valid for all t.

   False. Solving the DE:

   $$\int \frac{1}{1+y^2}\, dy = \int dt \quad \Rightarrow \quad \tan^{-1}(y) = t + C \quad \Rightarrow \quad y = \tan(t + c)$$

   The tangent function has vertical asymptotes at $t + c = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$, so there will be a strip of time of length π on which the solution will be valid (but no more).

10. We said earlier that, under certain circumstances, solution curves to $y' = f(t, y)$ cannot cross (or be tangent) in the direction field. What are the conditions?

    The conditions are those stated in the existence and uniqueness theorem (for uniqueness in particular). If $\partial f/\partial y$ is continuous, solutions cannot cross (or be tangent) in the direction field.

11. Show that, if $y' + p(t)y = q(t)y^n$, then dividing by y^n and subsituting $w = y^{1-n}$ gives you a linear DE (these were the Bernoulli equations).

    SOLUTION: Dividing by y^n, we have:

    $$\frac{y'}{y^n} + p(t)y^{1-n} = q(t)$$

    Making the substitution:

    $$w = y^{1-n} \qquad \Rightarrow w' = (1-n)\frac{y'}{y^n}$$

    we have:

    $$\frac{1}{1-n}w' + p(t)w = q(t)$$

    which is linear if we multiply by $1 - n$.

12. Problems 15-20 on page 8 (these are graphical questions- Be sure you look these over!).

2 Solve:

Give the general solution if there is no initial value. Before giving the solution, state what kind of differential equation it is (linear, separable, exact)- Multiple classes are possible, just give the one you will use.

1. $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{x}$

   Linear: $y' + \frac{2}{x}y = x$ Solve with an integrating factor of x^2 to get:

   $$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

2. $(x + y)\,dx - (x - y)\,dy = 0$. Homogeneous, so $v = y/x$, and

   $$\frac{dy}{dx} = \frac{x+y}{x-y} = \frac{1 + (y/x)}{1 - (y/x)} = \frac{1+v}{1-v}$$

   With the substitution $xv = y$, we get the substitution for dy/dx:

   $$v + xv' = y'$$

   So that the DE becomes:

   $$v + xv' = \frac{1+v}{1-v} \qquad \Rightarrow \qquad xv' = \frac{1+v}{1-v} - v = \frac{1 + v - v(1-v)}{1+v} = \frac{1+v^2}{1-v}$$

   The equation is now separable:

   $$\frac{1-v}{1+v^2}\,dv = \frac{1}{x}\,dx \qquad \Rightarrow \qquad \int \frac{1}{1+v^2}\,dv - \int \frac{v}{1+v^2}\,dv = \ln|x| + C$$

   Therefore,

   $$\tan^{-1}(v) - \frac{1}{2}\ln(1+v^2) = \ln|x| + C$$

   Lastly, back-substitute $v = y/x$.

3. $\dfrac{dy}{dx} = \dfrac{2x+y}{3+3y^2-x} \quad y(0)=0.$

   This is exact. The solution is, with $y(0)=0$,

$$-x^2 - xy + 3y + y^3 = 0$$

4. $\dfrac{dy}{dx} = -\dfrac{2xy+y^2+1}{x^2+2xy}$

   This is exact. The solution is: $x^2y + xy^2 + x = c$

5. $\frac{dy}{dt} = 2\cos(3t) \qquad y(0)=2$

   This is linear and separable. $y(t) = \frac{2}{3}\sin(3t) + 2$, and the solution is valid for all time.

6. $y' - \frac{1}{2}y = 0 \qquad y(0) = 200$. State the interval on which the solution is valid.

   This is linear and separable. The solution is $y(t) = 200\mathrm{e}^{(1/2)t}$

7. $y' - \frac{1}{2}y = \mathrm{e}^{2t} \qquad y(0) = 1$

   This is linear (but not separable). $y(t) = \frac{2}{3}\mathrm{e}^{2t} + \frac{1}{3}\mathrm{e}^{(1/2)t}$

8. $y' = \frac{1}{2}y(3-y)$

   Autonomous (and separable). Integrate using partial fractions:

$$\int \frac{1}{y(3-y)}\,dy = \frac{1}{2}\int\,dt$$

   Simplify your answer for y by dividing numerator and denominator appropriately to get:

$$y(t) = \frac{3}{(1/A)\mathrm{e}^{-(3/2)t} + 1}$$

9. $\sin(2t)\;dt + \cos(3y)\;dy = 0$

   Separable (and/or exact): $\frac{-1}{2}\cos(2t) + \frac{1}{3}\sin(3y) = C$

10. $t^2y' + 2ty - y^3 = 0$

    This is Bernoulli. Move the y^3 to the right then divide by it. We'll also divide by t^2:

$$\frac{y'}{y^3} + \frac{2}{t}\frac{1}{y^2} = \frac{1}{t^2}$$

    At this point, we see that the natural substitution will be

$$w = \frac{1}{y^2} \qquad \frac{dw}{dt} = -2\frac{y'}{y^3} \quad \text{or} \quad \frac{y'}{y^3} = -\frac{1}{2}w'$$

    Continuing, we put into a linear form to solve:

$$-\frac{1}{2}w' + \frac{2}{t}w = \frac{1}{t^2} \quad \Rightarrow \quad w' - \frac{4}{t}w = -\frac{2}{t^2} \quad \Rightarrow \quad \mathrm{e}^{\int p(t)\,dt} = \frac{1}{t^4}$$

    Therefore,

$$w = \frac{2}{5t} + Ct^4 \quad \Rightarrow \quad \frac{1}{y^2} = \frac{2}{5t} + Ct^4$$

    If the instructions say that you may leave your answer in implicit form, then this is fine. Otherwise, solve for y.

11. $y' = xy^2$

    Separable: $y = \dfrac{1}{-(1/2)x^2 - C}$

12. $\dfrac{dy}{dt} = \mathrm{e}^{t+y}$

Separable: $y' = \mathrm{e}^t \mathrm{e}^y$, so:

$$\int \mathrm{e}^{-y}\, dy = \int \mathrm{e}^t\, dt$$

and $-\mathrm{e}^{-y} = \mathrm{e}^t + C$

13. $\dfrac{dy}{dx} + y = \dfrac{1}{1+\mathrm{e}^x}$

Linear: $y' + y = 1/(1+\mathrm{e}^x)$, and the I.F. is e^x. Therefore,

$$(\mathrm{e}^x y) = \int \frac{\mathrm{e}^x}{1+\mathrm{e}^x}\, dx$$

To integrate, use u, du substitution. The solution is then:

$$y = \frac{\ln(1+\mathrm{e}^x) + C}{\mathrm{e}^x}$$

14. $(t^2 y + ty - y)\, dt + (t^2 y - 2t^2)\, dy = 0$

Does not seem to be exact. Try separating variables:

$$\frac{dy}{dt} = \frac{-y(t^2+t+1)}{(y-2)\cdot t^2}$$

so:

$$\frac{y-2}{y}\, dy = -\left(1 + \frac{1}{t} + t^{-2}\right)\, dt$$

(NOTE: Now the DE is also exact).

The solution is: $-y + 2\ln|y| = -\left(t + \ln|t| - \frac{1}{t} + C\right)$

15. $2xy^2 + 2y + (2x^2 y + 2x)y' = 0$

Exact. $x^2y^2 + 2xy = C$.

16. $x^3 \dfrac{dy}{dx} = 1 - 2x^2 y$.

Linear: $y' + \frac{2}{x} y = x^{-3}$, with integrating factor x^2:

$$y = \frac{\ln|x| + C}{x^2}$$

3 Misc.

1. Construct a linear first order differential equation whose general solution is given by:

$$y(t) = t - 3 + \frac{C}{t^2}$$

Construct y'. The idea will be to produce a linear DE. Therefore, we need to construct y' and compare it to y:

$$y' = 1 - 2Ct^{-3}$$

Add this to some multiple (t's are allowed) of y to get of the arbitrary constant. In this case,

$$y' + \frac{2}{t} y = (1 - 2Ct^{-3}) + 2 - \frac{6}{t} + 2Ct^{-3} = 3 - \frac{6}{t}$$

or,

$$ty' + 2y = 3t - 6$$

2. Construct a linear first order differential equation whose general solution is given by:

$$y(t) = 2\sin(3t) + C\mathrm{e}^{-2t}$$

$$y' = 6\cos(3t) - 2C\mathrm{e}^{-2t}$$

so that: $y' + 2y = 4\sin(3t) + 6\cos(3t)$.

3. Construct an autonomous differential equation that has stable equilibria at $y(t) = 1$ and $y(t) = 3$, and one unstable equilibrium at $y(t) = 2$. (Hint: Draw the phase plot first).

The formula would be something like:

$$y' = -\alpha(y-1)(y-2)(y-3)$$

with $\alpha > 0$.

4. Construct an autonomous differential equation so that all solutions tend towards $y(t) = 2$ as $t \to \infty$.

(Compare to the quiz question). Using a phase plot, we see that any line with a negative slope and going through $(2, 0)$ will work for the DE. You could get more exotic with a shifted cubic if you wanted to, but we'll stay simple:

$$y' = m(y-2), \qquad m < 0$$

5. Suppose we have a tank that contains M gallons of water, in which there is Q_0 pounds of salt. Liquid is pouring into the tank at a concentration of r pounds per gallon, and at a rate of γ gallons per minute. The well mixed solution leaves the tank at a rate of γ gallons per minute.

Write the initial value problem that describes the amount of salt in the tank at time t, and solve:

$$\frac{dQ}{dt} = r\gamma - \frac{\gamma}{M}Q, \qquad Q(0) = Q_0$$

The solution is:

$$Q = rM + (Q_0 - rM)\mathrm{e}^{-(\gamma/M)t}$$

6. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on Q_0? Does this make sense?

Note that the differential equation for Q is autonomous, so we could do a phase plot (line with a negative slope). Or, we can just take the limit as $t \to \infty$ and see that $Q \to rM$. This does not necessarily depend on Q_0; if Q_0 starts at equilibrium, rM, then Q is constant.

It does make sense. The incoming concentration of salt is r pounds per gallon, so we would expect the long term concentration to be the same, $rM/M = r$.

7. Modify problem 5 if: $M = 100$ gallons, $r = 2$ and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if $Q_0 = 50$.

$$\frac{dQ}{dt} = 4 - \frac{3}{100+t}Q \qquad Q(0) = 50$$

This goes from being autonomous to linear. In this case, use an integrating factor,

$$\mathrm{e}^{\int p(t)\,dt} = \mathrm{e}^{3\int \frac{1}{100+t}\,dt} = \mathrm{e}^{3\ln|100+t|} = (100+t)^3 \qquad t > -100$$

Continuing, we get:

$$Q(t) = -\frac{50,000,000}{(100+t)^3} + 100 + t$$

8. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v$, find the initial value problem (and solve it) for the velocity at time t.

   The general model is: $mv' = mg - kv$. In this case, $m = 1$, $g = 9.8$ and $k = 1/2$. Therefore,

$$v' = 9.8 - \frac{1}{2}v$$

   Which is linear (and autonomous). Since the object is being dropped, the initial velocity is zero.

   Solve it:

$$v(t) = 19.6\left(1 - \mathrm{e}^{-(1/2)t}\right)$$

9. (Continuing with the last problem): At $t = 10$ minutes, the force due to air resistance suddenly changes to $10v$. Model the velocity for $t \geq 10$ (set up and solve the IVP):

   The dynamics are now:

$$v' = 9.8 - 10v$$

   In order to make v continuous, the initial condition used here will be where the velocity left off after the last problem.

   If we make time re-start at zero (so that t is minutes after the previous 10), we would make $v(0) = 19.6\left(1 - \mathrm{e}^{-5}\right)$, which is approximately 19.467. The solution for t minutes after the original 10 minutes is:

$$v(t) = 0.98 + 18.487\mathrm{e}^{-10t}$$

   NOTE: If you did not restart time, the initial condition would be the same, except $v(10) \approx 19.467$, and the solution would be scaled:

$$v(t) = 0.98 + (18.487 \times \mathrm{e}^{100})\mathrm{e}^{-10t}$$

   valid for $t > 10$.

10. (Continuing with the falling object): In a direction field, draw a sketch of the solution. HINT: These are autonomous differential equations, so you should draw the phase plot first!

    To draw the direction field, first get the equilibrium solutions:

$$v = 2 \cdot 9.8 = 19.6 \qquad v = \frac{9.8}{10} = 0.98$$

    Your direction field should look something like Figure 1.

11. Suppose $y' = ky(1 - y)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

    Your phase plot should be an upside down parabola intersecting with $y = 0$ and $y = 1$. We see that $y = 0$ is unstable and $y = 1$ is stable.

    (a) $y(t)$ is increasing if $0 < y_0 < 1$, decreasing otherwise.

    (b) For concavity, recall that, if $y' = f(y)$, then:

$$\frac{d^2y}{dt^2} = \frac{df}{dy} \cdot \frac{dy}{dt} = \frac{df}{dy} \cdot f(y)$$

    Now, if $y < 0$, we see that $df/dy > 0$ and $f(y) < 0$, so y is concave down and decreasing.
    If $0 < y < 1/2$, then $df/dy > 0$ and $f(y) > 0$, so y is concave up and increasing.
    If $1/2 < y < 1$, then $df/dy < 0$ and $f(y) > 0$, so y is concave down and increasing.
    If $y > 1$, then $df/dy < 0$ and $f(y) < 0$, so y is concave up and decreasing.

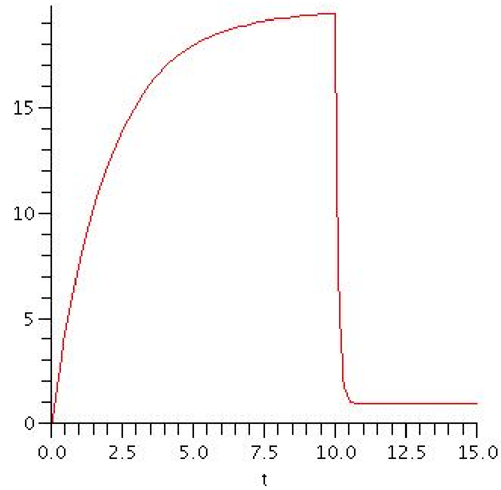

Figure 1: A graph of the velocity for the falling object problem. Note where the equilibria are: $v = 19.6$ for $t < 10$ and $v = 0.98$ for $t > 10$.

(c) The analytic solution (you'll need integration by parts):

$$\int \frac{1}{y(1-y)}\,dy = \int k\,dt \quad \Rightarrow \quad \int \frac{1}{y} + \frac{1}{1-y}\,dy = kt + C \quad \Rightarrow \quad \ln|y| - \ln|1-y| = kt + C$$

Now,

$$\ln\left|\frac{y}{1-y}\right| = kt + C \quad \Rightarrow \quad \frac{y}{1-y} = A\mathrm{e}^{kt} \quad \Rightarrow \quad y(t) = \frac{A\mathrm{e}^{kt}}{1 + A\mathrm{e}^{kt}}$$

In class, we simplified this to:

$$y(t) = \frac{1}{(1/A)\mathrm{e}^{-kt} + 1}$$

which is easier to analyze.