Selected Solutions, Section 5.2

For problems 2, 5, 6, 8 do not spend too much time finding the general term(s) of the series. The recursion relationships are typically as far as we'll need to go.
In each of these problems, we take:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

2. In this case,

$$y'' - xy' - y = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

Notice that the middle sum begins with x^1 rather than a constant (as the other sums do). We could simply begin that index with zero, write the first sum to match the other indices, then collect terms: Let $k = n - 2$, or $n = k + 2$ and:

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$$

Now,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k - \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n =$$

$$\sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} - n a_n - a_n\right) x^n = \sum_{n=0}^{\infty} \left((n+2)(n+1) a_{n+2} - (n+1) a_n\right) x^n$$

This gives us the recursion relation:

$$(n+2)(n+1) a_{n+2} - (n+1) a_n = 0 \quad \Rightarrow \quad (n+2)(n+1) a_{n+2} = (n+1) a_n$$

$$a_{n+2} = \frac{a_n}{n+2}$$

Notice that

$$a_2 = \frac{1}{2} a_0 \quad a_3 = \frac{1}{3} a_1 \quad a_4 = \frac{1}{4} a_2 = \frac{1}{4 \cdot 2} a_0 \quad a_5 = \frac{1}{5} a_3 = \frac{1}{5 \cdot 3} a_1$$

$$a_6 = \frac{1}{6} a_4 = \frac{1}{6 \cdot 4 \cdot 2} a_0 \quad a_7 = \frac{1}{7} a_5 = \frac{1}{7 \cdot 5 \cdot 3} a_1 \quad a_8 = \frac{1}{8} a_6 = \frac{1}{8 \cdot 6 \cdot 4 \cdot 2} a_0$$

and so on. We can write the solution $y(x)$ as:

$$y(x) = a_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{4 \cdot 2} x^4 + \frac{1}{6 \cdot 4 \cdot 2} x^6 + \ldots\right) + a_1 \left(x + \frac{1}{3} x^3 + \frac{1}{5 \cdot 3} x^5 + \frac{1}{7 \cdot 5 \cdot 3} x^7 + \ldots\right)$$

These two functions (in series form) make up our fundamental set.

5. Follows much the same procedure:

$$(1-x)y'' + y = (1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n$$

Multiply by the $1-x$ and incorporate the x into the sum:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n$$

Now our difficulty is that the middle sum begins with x^1 but the others do not (beginning with $n=1$ would fix it). Also, the indices do not currently match. Start every sum with the same thing:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} = \sum_{k=0}^{\infty}(k+1)ka_{k+1}x^k$$

Now we can write the differential equation as a single power series:

$$\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n+1)a_{n+1} + a_n\right]x^n$$

Giving us the recursion relation:

$$a_{n+2} = \frac{n}{n+2}a_{n+1} - \frac{1}{(n+2)(n+1)}a_n$$

In general, this means:

$$\begin{array}{lcll} a_2 & = & & -\frac{1}{2}a_0 \\ a_3 & = & \frac{1}{3}a_2 & -\frac{1}{6}a_1 \\ a_4 & = & \frac{1}{2}a_3 & -\frac{1}{12}a_2 \\ a_5 & = & \frac{3}{5}a_4 & -\frac{1}{20}a_3 \end{array}$$

and so on. To get our fundamental set, solve these first with $a_0 = 1, a_1 = 0$, then with $a_0 = 0$, $a_1 = 1$:

$$\begin{array}{lcl} a_2 & = & -\frac{1}{2} \\ a_3 & = & -\frac{1}{6} \\ a_4 & = & -\frac{1}{24} \end{array} \qquad \begin{array}{lcl} a_2 & = & 0 \\ a_3 & = & -\frac{1}{6} \\ a_4 & = & -\frac{1}{12} \\ a_5 & = & -\frac{1}{24} \end{array}$$

$$y(x) = a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots\right) + a_1\left(x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 + \dots\right)$$

6. Goes much the same as Problem 5. Be sure to get your sums to all match in terms of beginning power of x and the index.

$$(2+x^2)y'' - xy' + 4y = (2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 4a_n x^n$$

We'll try to manipulate the first sum to look like the second two:

$$(2+x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

The second sum begins with x^2, but beginning with zero might be OK, since the first two terms would be zero. Shift the index of the first sum to match ($k = n-2$ or $n = k+2$):

$$\sum_{k=0}^{\infty} 2(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} k(k-1)a_k x^k$$

Now we can collect all the terms together:

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + n(n-1)a_n - na_n + 4a_n\right] x^k =$$

$$\sum_{n=0}^{\infty} \left[2(n+2)(n+1)a_{n+2} + (n^2-2n+4)a_n\right] x^k$$

This gives us the recursion:

$$a_{n+2} = -\frac{n^2-2n+4}{2(n+2)(n+1)}a_n$$

or:

$$a_2 = -a_0 \quad a_3 = -\frac{1}{4}a_1 \quad a_4 = -\frac{1}{6}a_2 = \frac{1}{6}a_0$$

$$a_5 = -\frac{7}{40}a_3 = \frac{7}{160}a_1 \qquad a_6 = -\frac{1}{5}a_4 = -\frac{1}{30}a_0$$

and so on. Writing y in terms of its fundamental set,

$$y(x) = a_0\left(1 - x^2 + \frac{1}{6}x^4 - \frac{1}{30}x^6 + \ldots\right) + a_1\left(x - \frac{1}{4}x^3 + \frac{7}{160}x^5 + \ldots\right)$$

8. Be careful in that our power series is now based at $x_0 = 1$ instead of $x_0 = 0$:

$$xy'' + y' + xy = x\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=1}^{\infty} na_n(x-1)^{n-1} + x\sum_{n=0}^{\infty} a_n(x-1)^n$$

The problem is that we cannot incorporate x into a series with an $(x-1)$ expansion. However, note that we can write

$$x = x - 1 + 1 \qquad \text{or } x = 1 + (x-1)$$

Making this substitution into the first sum,

$$(1+(x-1))\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}$$

And similarly, into the last sum:

$$(1+(x-1))\sum_{n=0}^{\infty} a_n(x-1)^n = \sum_{n=0}^{\infty} a_n(x-1)^n + \sum_{n=0}^{\infty} a_n(x-1)^{n+1}$$

Notice that our usual shift in the index won't work here- 2 of the sums start with $(x-1)^1$, the other 3 with constants. We will factor the constants out, and begin all indices at $n = 1$. Here are the five sums:

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} = 2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n$$

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1} = \sum_{n=1}^{\infty}(n+1)na_{n+1}(x-1)^n$$

$$\sum_{n=1}^{\infty} na_n(x-1)^{n-1} = a_1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}(x-1)^n$$

$$\sum_{n=0}^{\infty} a_n(x-1)^n = a_0 + \sum_{n=1}^{\infty} a_n(x-1)^n$$

$$\sum_{n=0}^{\infty} a_n(x-1)^{n+1} = \sum_{n=1}^{\infty} a_{n-1}(x-1)^n$$

Now simply collect it all into a single sum and extract the recursion:

$$(2a_2 + a_1 + a_0) +$$

$$\sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + (n+1)a_{n+1} + a_n + a_{n-1}\right](x-1)^n$$

with recursion:

$$a_{n+2} = -\frac{(n+1)^2 a_{n+1} + a_n + a_{n-1}}{(n+2)(n-1)} \qquad n = 1, 2, 3, \ldots$$

To get our fundamental set, we would first set $a_0 = 1$, $a_1 = 0$. We could then compute the coefficients to get $y_1(x)$. Next, set $a_0 = 0$, $a_1 = 1$ to get y_2 by computing the coefficients from our recursion.

15, 16 (We'll do these in Maple later)

19. Optional.

20. Good practice with the Ratio Test.