Help on section 6.2

2. Table entry 11.

4.
$$\frac{3s}{s^2-s-6}=\frac{A}{s+2}+\frac{B}{s-3}$$
so that
$$A(s-3)+B(s+2)=3s$$
If $s=3$, then $B=9/5$. If $s=-2$, then $A=6/5$.

5.
$$\frac{2s+2}{s^2+2s+5}=\frac{2s+2}{(s^2+2s+1)+5-1}=\frac{2s+2}{(s+1)^2+2^2}=\frac{2(s+1)}{(s+1)^2+2^2}$$

8. Partial fractions:
$$\frac{8s^2-4s+12}{s(s^2+4)}=\frac{A}{s}+\frac{Bs+C}{s^2+4}$$
Therefore,
$$8s^2-4s+12=A(s^2+4)+(Bs+C)s \quad\Rightarrow\quad 8s^2-4s+12=(A+B)s^2+Cs+4A$$
Therefore,
$$\begin{aligned} A+B &= 4\\ C &= -4\\ 4A &= 12\end{aligned} \quad\Rightarrow\quad A=3, C=-4, B=1$$
And we get:
$$\frac{8s^2-4s+12}{s(s^2+4)}=\frac{3}{s}+\frac{5s-4}{s^2+4}$$

10. Complete the square in the denominator:
$$\frac{2s-3}{s^2+2s+1+9}=\frac{2(s+1)-5}{(s+1)^2+9}=2\frac{s+1}{(s+1)^2+3^2}-\frac{5}{3}\frac{3}{(s+1)^2+3^2}$$

12. Take the Laplace transform of both sides:
$$s^2Y-sy(0)-y'(0)+3(sY-y(0))+2Y=0$$
Apply initial conditions and solve for $Y(s)$:
$$Y(s)=\frac{s+3}{s^2+3s+2}=\frac{2}{s+1}+\frac{1}{s+2}$$
so that
$$y(t)=2\mathrm{e}^{-t}-\mathrm{e}^{-2t}$$

14. Same idea:

$$s^2Y - sy(0) - y'(0) - 4(sY - y(0)) + 4Y = 0$$

Apply initial conditions and solve for $Y(s)$:

$$Y(s) = \frac{s-3}{(s^2-4s+4} = \frac{(s-3)}{(s-2)^2} = \frac{(s-2)-1}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

Use table entries 2 and 11:

$$y(t) = \mathrm{e}^{2t} - t\mathrm{e}^{2t}$$

20. Continuing in the same fashion as before, you should find that

$$Y(s) = \frac{s}{(s^2+\omega^2)(s^2+4)} + \frac{s}{s^2+\omega^2}$$

Use Partial Fractions on the first term:

$$\frac{s}{(s^2+\omega^2)(s^2+4)} = \frac{As+B}{s^2+\omega^2} + \frac{Cs+D}{s^2+4}$$

so that (multiply it out and equate coefficients):

$$\begin{array}{c|rl} s^3 & A+C & =0 \\ s^2 & B+D & =0 \\ s & 4A+C\omega^2 & =1 \\ \text{const} & 4B+D\omega^2 & =0 \end{array} \quad \Rightarrow \quad A = \frac{1}{4-\omega^2}, C=-A, B=0, D=0$$

22. Same idea as before.

$$Y(s) = \frac{1}{(s+1)(s^2-2s+2)} + \frac{1}{s^2-2s+2}$$

Using partial fractions again, get that:

$$Y(s) = \frac{1/5}{s+1} - \frac{1}{5}\cdot\frac{s-3}{s^2-2s+2} + \frac{1}{s^2-2s+2}$$

Now complete the square in the denominator of the last two terms.

24. For the Laplace transform of the right hand side of the given differential equation, define $f(t)$ as the piecewise defined function. Then:

$$\int_0^\infty \mathrm{e}^{-st} f(t)\,dt = \int_0^\pi \mathrm{e}^{-st}\,dt + \int_\pi^\infty 0\,dt = -\frac{\mathrm{e}^{-st}}{s}\bigg|_0^\pi = \frac{1-\mathrm{e}^{-\pi s}}{s}$$

Therefore, the Laplace transform will be:

$$Y(s) = \frac{1-\mathrm{e}^{-\pi s}}{s(s^2+4)} + \frac{s}{s^2+4}$$

28.

$$F'(s) = \frac{d}{ds}\left(\int_0^\infty \mathrm{e}^{-st} f(t)\,dt\right) = \int_0^\infty \frac{d}{ds}\left(\mathrm{e}^{-st}\right) f(t)\,dt = \int_0^\infty -t\mathrm{e}^{-st} f(t)\,dt = \mathcal{L}(-tf(t))$$

29. Use Exercise 28, which in this case is

$$-\mathcal{L}(-t\mathrm{e}^{at}) = -F'(s)$$

where $F(s) = \frac{1}{s-a}$. Therefore,

$$-\mathcal{L}(-t\mathrm{e}^{at}) = \frac{1}{(s-a)^2}$$

As in table entry 11 with $n = 1$.

30. Same idea as Exercise 29, except we differentiate twice (a little messy) to get:

$$\mathcal{L}(t^2\sin(bt)) = \frac{d^2}{ds^2}\left(\frac{b}{s^2+b^2}\right) = \frac{d}{ds}\left((-2bs)(s^2+b^2)^{-2}\right) = \frac{6bs^2-2b^3}{(s^2+b^2)^3}$$

33. Differentiate table entry 9 with respect to s, then multiply by -1.

37. This one is a bit tricky (A Challenge Problem!)

$$\mathcal{L}(g(t)) = \int_0^\infty \mathrm{e}^{-st}\left[\int_0^t f(w)\,dw\right]\,dt = \int_0^\infty \int_0^t \mathrm{e}^{-st} f(w)\,dw\,dt$$

In the (t, w) plane, we have $0 \le w \le t$ and $0 \le t \le \infty$. If we reverse the order of integration then $t \ge w$ and $0 \le w \le \infty$ (draw it in the (t, w) plane). Therefore,

$$\int_0^\infty f(w)\left[\int_w^\infty \mathrm{e}^{-st}\right]\,dt\,dw = \int_0^\infty f(w)\frac{\mathrm{e}^{-sw}}{s}\,dw$$

which is what we wanted: $\frac{1}{s}\mathcal{L}(f(t))$.