Section 6.4 Homework Notes

2.

$$y'' + 2y' + 2y = u_\pi(t) - u_{2\pi}(t), \quad y(0) = 0, \quad y'(0) = 1$$

Take the Laplace transforms and solve for $Y(s)$:

$$(s^2 Y - 0 - 1) + 2(sY - 0) + 2Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s}$$

$$(s^2 + 2s + 2)Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s} + 1$$

$$Y = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s(s^2 + 2s + 2)} + \frac{1}{s^2 + 2s + 2}$$

We'll do the last term first:

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{s^2 + 2s + 1 + 1} = \frac{1}{(s+1)^2 + 1}$$

so the inverse Laplace transform is (table entry 19): $\mathrm{e}^{-t}\sin(t)$.

Next, notice that the first term is of the form:

$$\left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)\frac{1}{s(s^2 + 2s + 2)} = \left(\mathrm{e}^{-\pi s} - \mathrm{e}^{-2\pi s}\right)H(s)$$

So if we find $h(t)$, the inverse Laplace transform of this part will be:

$$u_\pi(t)h(t - \pi) - u_{2\pi}(t)h(t - 2\pi)$$

Therefore, we only need to focus on inverting:

$$H(s) = \frac{1}{s(s^2 + 2s + 2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 2}$$

Solving, we get $A = 1/2$, $B = -1/2$, $C = -1$, or:

$$\frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s+2}{s^2 + 2s + 2} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\left[\frac{s+1}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1}\right]$$

Now $h(t) = \frac{1}{2} - \frac{1}{2}\left(\mathrm{e}^{-t}\cos(t) + \mathrm{e}^{-t}\sin(t)\right)$. Putting it all together,

$$y(t) = \mathrm{e}^{-t}\sin(t) + u_\pi(t)h(t - \pi) - u_{2\pi}(t)h(t - 2\pi)$$

In Maple, here are the commands:

```
with(plots)
Eqn02:=diff(y(t),t$2)+2*diff(y(t),t)+2*y(t)=Heaviside(t-Pi)-Heaviside(t-2*Pi);
Y1:=dsolve({Eqn02,y(0)=0,D(y)(0)=1},y(t));
A:=plot(rhs(Y1),t=0..9,color=red);
B:=plot(Heaviside(t-Pi)-Heaviside(t-2*Pi),t=0..9, color=green);
display(A,B);
```

4. Take the Laplace transform of both sides to get:

$$(s^2 + 4)Y = \frac{1}{s^2 + 1} + \frac{\mathrm{e}^{-\pi s}}{s^2 + 1} \quad \Rightarrow \quad Y = \frac{1 + \mathrm{e}^{-\pi s}}{(s^2 + 1)(s^2 + 4)}$$

Using Partial Fractions, we note that:

$$\frac{1}{(s^2 + 1)(s^2 + 4)} = \frac{As + B}{s^2 + 1} + \frac{Cs + D}{s^2 + 4} \quad \Rightarrow \quad A = -\frac{1}{3} \qquad C = \frac{1}{3} \quad B = 0, D = 0$$

(and continue).

The Maple code for this one is:

```
with(plots):
Eqn04:=diff(y(t),t$2)+4*y(t)=sin(t)+Heaviside(t-Pi)*sin(t-Pi);
Y1:=dsolve({Eqn04,y(0)=0,D(y)(0)=0},y(t));
A:=plot(rhs(Y1),t=0..9,color=red);
B:=plot(sin(t)-Heaviside(t-Pi)*sin(t-Pi),t=0..9, color=green);
display(A,B);
```

6.

$$y'' + 3y' + 2y = u_2(t), \qquad y(0) = 0, \quad y'(0) = 1$$

As is usual, take the Laplace transforms and solve for $Y(s)$:

$$(s^2Y - 0 - 1) + 3(sY - 0) + 2Y = \mathrm{e}^{-2s}\frac{1}{s}$$

$$Y = \mathrm{e}^{-2s}\frac{1}{s(s^2+3s+2)} + \frac{1}{s^2+3s+2}$$

Break it up- Let's do the second term first:

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1} = -\frac{1}{s+2} + \frac{1}{s+1}$$

so the inverse Laplace transform of this part is: $-\mathrm{e}^{-2t} + \mathrm{e}^{-t}$.

Alternative Solution to this part. We could have completed the square in the denominator:

$$\frac{1}{s^2+3s+2} = \frac{1}{s^2+3s+\frac{9}{4}+2-\frac{9}{4}} = 2\frac{\frac{1}{2}}{\left(s+\frac{3}{2}\right)^2 - \frac{1}{4}}$$

Combine table entries 14 and 7 to get the inverse Laplace transform as:

$$2\mathrm{e}^{-\frac{3}{2}t}\sinh\left(\frac{1}{2}t\right)$$

This is the solution that Maple gives you. Notice that it is the same as our solution:

$$2\mathrm{e}^{-\frac{3}{2}t}\sinh\left(\frac{1}{2}t\right) = 2\mathrm{e}^{-\frac{3}{2}t}\cdot\frac{1}{2}\left(\mathrm{e}^{\frac{1}{2}t} - \mathrm{e}^{-\frac{1}{2}t}\right) = \mathrm{e}^{-t} - \mathrm{e}^{-2t}$$

It is probably easier to factor and use partial fractions... Here is the Maple code:

```
with(plots):
Eqn06:=diff(y(t),t$2)+3*diff(y(t),t)+2*y(t)=Heaviside(t-2);
Y1:=dsolve({Eqn06,y(0)=0,D(y)(0)=1},y(t));
A:=plot(rhs(Y1),t=0..9,color=red);
B:=plot(Heaviside(t-2),t=0..9, color=green);
display(A,B);
```

8. Same idea as before. In this case, the algebra gets a bit messy. Here is the partial fraction decomposition:

$$\frac{1}{s^2(s^2+s+\frac{5}{4})} = \frac{4}{5}\cdot\frac{1}{s^2} - \frac{16}{25}\cdot\frac{1}{s} + \frac{4}{25}\frac{4s-1}{s^2+s+5/4}$$

This is a good problem for Maple! Here is the code- You only need to type `with(plots):` once somewhere on the worksheet, so if you've already done it, you can skip that line:

```
with(plots):
Eqn08:=diff(y(t),t$2)+diff(y(t),t)+(5/4)*y(t)=t-Heaviside(t-Pi/2)*(t-Pi/2);
Y1:=dsolve({Eqn08,y(0)=0,D(y)(0)=0},y(t));
A:=plot(rhs(Y1),t=0..9,color=red);
B:=plot(t-(t-Pi/2)*Heaviside(t-Pi/2),t=0..9, color=green);
display(A,B);
```

14. We want a function g that ramps up from the point $(t_0, 0)$ to $(t_0 + k, h)$ and then stays at that value. The line segment has the equation: $y = \frac{h}{k}(t - t_0)$, so overall,

$$f(t) = \frac{h}{k}(t - t_0)(u_{t_0}(t) - u_{t_0+k}(t)) + \frac{h}{k}u_{t_0+k}$$

15. Starts the same as 14, but ramps back down to zero:

- Ramps up from $(t_0, 0)$ to $(t_0 + k, h)$.
- Then ramps down from $(t_0 + k, h)$ to $(t_0 + 2k, 0)$.

Notice that this is simply two line segments. We have pairs of points for each, so let's write the equation of each line segment:

- Slope: $\frac{h}{k}$, so: $y - 0 = \frac{h}{k}(t - t_0)$.
- Slope: $\frac{-h}{k}$, so: $y - 0 = -\frac{h}{k}(t - (t_0 + 2k))$

Finally, use the "On-Off" switch for each line segment. The first line segment comes on at time t_0, off at $t_0 + k$):

$$(u_{t_0}(t) - u_{t_0+k}(t))\left(\frac{h}{k}(t - t_0)\right)$$

We'll add in the second switch,

$$(u_{t_0+k}(t) - u_{t_0+2k}(t))\left(-\frac{h}{k}(t - t_0 - 2k)\right)$$

Add everything together. To get the answer in the text, factor the slope out and expand (for us, you can leave your answer in this form).

$$(u_{t_0}(t) - u_{t_0+k}(t))\left(\frac{h}{k}(t - t_0)\right) + (u_{t_0+k}(t) - u_{t_0+2k}(t))\left(-\frac{h}{k}(t - t_0 - 2k)\right)$$

Alternatively, you could also have written the second line segment using the first ordered pair.

16. It is a little heavy on the algebra- Let's see how we do. With zero IC's, the Laplace transform is simple to construct:

$$(s^2 + \frac{1}{4}s + 1)Y = \frac{k}{s}(\mathrm{e}^{-3/2s} - \mathrm{e}^{-5/2s})$$

so that

$$Y = \frac{k}{s(s^2 + \frac{1}{4}s + 1)}(\mathrm{e}^{-3/2s} - \mathrm{e}^{-5/2s})$$

Now the partial fraction decomposition is:

$$H(s) = \frac{1}{s(s^2 + \frac{1}{4}s + 1)} = \frac{1}{s} - \frac{s + 1/4}{s^2 + \frac{1}{4}s + 1} = \frac{1}{s} - \frac{s + 1/8}{(s + \frac{1}{8})^2 + \frac{63}{64}} - \frac{1}{8} \cdot \frac{8}{\sqrt{63}} \frac{\sqrt{63/64}}{(s + \frac{1}{8})^2 + \frac{63}{64}}$$

so that

$$h(t) = 1 - \mathrm{e}^{-t/8}\cos\left(\frac{3\sqrt{7}}{8}t\right) - \frac{1}{\sqrt{63}}\mathrm{e}^{-t/8}\sin\left(\frac{3\sqrt{7}}{8}t\right)$$

The overall solution is therefore: $ku_{3/2}(t)h(t-3/2) - ku_{5/2}(t)h(t-5/2)$.

For the rest, Maple is required. Here is the code to help with (d), and part (e) is similarly solved by using the graphics in Maple. Part (d) is answered via a small animation. (For fun, you can convert Maple animation into an (animated) GIF file to post on the web! To do this, right-click on the animation, choose Export, choose GIF, then save. To see the animation, open it using Firefox or any web browser).

```
Eqn16:=diff(y(t),t$2)+(1/4)*diff(y(t),t)+y(t)=k*Heaviside(t-3/2)-k*Heaviside(t-5/2);
Y1:=dsolve({Eqn16,y(0)=0,D(y)(0)=0},y(t));
animate(plot,[rhs(Y1)-2,t=0..9], k=2..3);
```

19. Straightforward to write down, a little tricky to analyze:

$$y'' + y = u_0(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi}(t)$$

Taking the Laplace transform, solve for Y and inverting:

$$(s^2+1)Y = \frac{1}{s} + 2\sum_{k=1}^{n}(-1)^k \mathrm{e}^{-k\pi s}\frac{1}{s} \quad \Rightarrow \quad Y = \frac{1}{s(s^2+1)} + 2\sum_{k=1}^{n}(-1)^k \mathrm{e}^{-k\pi s}\frac{1}{s(s^2+1)}$$

where

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

so that

$$y(t) = 1 - \cos(t) + 2\sum_{k=1}^{n}(-1)^k u_{k\pi t}\left(1 - \cos(t - k\pi)\right)$$

This is a bit difficult to analyze in this form. However, if we consider the graph of the cosine, we see that:

If k is odd $\quad \cos(t-k\pi) = -\cos(t)$
If k is even $\quad \cos(t-k\pi) = \cos(t)$

Here we write out the function in the form of a table:

kth term		Term is active:
$k=1$	$-2(1+\cos(t))$	$t \geq \pi$
$k=2$	$2(1-\cos(t))$	$t \geq 2\pi$
$k=3$	$-2(1+\cos(t))$	$t \geq 3\pi$
$k=4$	$2(1-\cos(t))$	$t \geq 4\pi$
$\vdots$	$\vdots$	

Therefore, writing $y(t)$ in piecewise form (for clarity, I'm writing it as a table):

Interval	$y(t)$
$t < \pi$	$-\cos(t) + 1$
$\pi \leq t < 2\pi$	$(1-\cos(t) - 2(1+\cos(t)) = -3\cos(t) - 1$
$2\pi \leq t < 3\pi$	$3(1-\cos(t)) - 2(1+\cos(t)) = -5\cos(t) + 1$
$3\pi \leq t < 4\pi$	$3(1-\cos(t)) - 4(1+\cos(t)) = -7\cos(t) - 1$
$4\pi \leq t < 5\pi$	$5(1-\cos(t)) - 4(1+\cos(t)) = -9\cos(t) + 1$

Here is the Maple code to help (we'll use the `sum` command!)

```
N:=15;
Eqn19:=diff(y(t),t$2)+y(t)=Heaviside(t)+2*sum((-1)^k*Heaviside(t-k*Pi),k=1..N);
Y1:=dsolve({Eqn19,y(0)=0,D(y)(0)=0},y(t),method=laplace);
```

```
plot(rhs(Y1),t=0..20*Pi);
N:=30;
Eqn19:=diff(y(t),t$2)+y(t)=Heaviside(t)+2*sum((-1)^k*Heaviside(t-k*Pi),k=1..N);
Y1:=dsolve({Eqn19,y(0)=0,D(y)(0)=0},y(t),method=laplace);
plot(rhs(Y1),t=0..50*Pi);
```

20. *Addendum: Maple seems to break down solving 20...* Here is some Maple code to force Maple to use the sums. This is a nice example because we're solving it step-by-step!

```
with(inttrans):
N:=5;
Eqn20:=diff(y(t),t$2)+(1/10)*diff(y(t),t)+y(t)=Heaviside(t)+2*Sum((-1)^k*Heaviside(t-k*Pi),k=1..N);
Y1:=laplace(Eqn20,t,s);
Y2:=subs(y(0)=0,D(y)(0)=0,Y1);
Y3:=solve(Y2,laplace(y(t),t,s));
Y4:=invlaplace(Y3,s,t);
plot(Y4,t=0..100);
```

Same technique should work for 21 and 22.