Complex Numbers
A Helpful Trick for Integration

Worked Example:

Use the fact that $\int e^{(a+ib)t}\,dt = \frac{1}{a+ib}e^{(a+ib)t}$ to compute $\int e^{2t}\cos(3t)\,dt$
SOLUTION:
We note that

$$e^{(2+3i)t} = e^{2t}\left(\cos(3t) + i\sin(3t)\right)$$

Therefore,

$$\int e^{(2+3i)t}\,dt = \int e^{2t}\left(\cos(3t) + i\sin(3t)\right)\,dt = \int e^{2t}\cos(3t)\,dt + i\int e^{2t}\sin(3t)\,dt$$

Therefore, the desired integral is the real part of the antiderivative of the exponential:

$$\int e^{(2+3i)t}\,dt = \frac{1}{2+3i}e^{(2+3i)t} = \frac{2-3i}{13}e^{2t}(\cos(3t) + i\sin(3t)) =$$

$$e^{2t}\left(\frac{2}{13} - \frac{3}{13}i\right)(\cos(3t) + i\sin(3t)) =$$

$$e^{2t}\left[\left(\frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)\right) + i\left(-\frac{3}{13}\cos(3t) + \frac{2}{13}\sin(3t)\right)\right]$$

Taking the real part, we get our answer:

$$\int e^{2t}\cos(3t)\,dt = e^{2t}\left(\frac{2}{13}\cos(3t) + \frac{3}{13}\sin(3t)\right)$$

In fact, we get the other integral for free:

$$\int e^{2t}\sin(3t)\,dt = e^{2t}\left(\frac{-3}{13}\cos(3t) + \frac{2}{13}\sin(3t)\right)$$

Homework Addition to Section 6.1

1. Use e^{iat} to help compute the Laplace transform of $\cos(at)$ (compare to Exercise 6)

2. Use $e^{(a+ib)t}$ to help compute the Laplace transform of $e^{at}\sin(bt)$ and $e^{at}\cos(bt)$ (compare to exercises 13, 14).

3. If the function is of exponential order, find the M, k from the definition. Otherwise, state that it is not of exponential order.

   (a) $\sin(t)$
   (b) $\tan(t)$
   (c) t^3
   (d) e^{t^2}
   (e) 5^t
   (f) t^t