Solutions to Review Questions: Exam 3

(NOTE: The exam will not cover Section 5.3)

1. Definitions: The Heaviside function is defined as:

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases} \qquad c > 0$$

The Dirac δ-function is defined as:

$$\delta(t-c) = \lim_{\tau \to 0} d_\tau(t-c)$$

where

$$d_\tau(t-c) = \begin{cases} \frac{1}{2\tau} & \text{if } c - \tau < t < c + \tau \\ 0 & \text{elsewhere} \end{cases}$$

The convolution operation is defined as:

$$(f * g)(t) = \int_0^t f(t-u)g(u)\,du$$

2. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

(a)

$$f(t) = \begin{cases} 3, & 0 \leq t < 2 \\ 6 - t, & t \geq 2 \end{cases}$$

$$\int_0^\infty \mathrm{e}^{-st} f(t)\,dt = \int_0^2 3\mathrm{e}^{-st}\,dt + \int_2^\infty (6-t)\mathrm{e}^{-st}\,dt$$

The second antiderivative is found by integration by parts:

$$\int_2^\infty (6-t)\mathrm{e}^{-st}\,dt \Rightarrow \begin{array}{ccc} \hline + & 6-t & \mathrm{e}^{-st} \\ - & -1 & (-1/s)\mathrm{e}^{-st} \\ + & 0 & (1/s^2)\mathrm{e}^{-st} \end{array} \Rightarrow \quad \mathrm{e}^{-st}\left(-\frac{6-t}{s} + \frac{1}{s^2}\right)\Big|_2^\infty$$

Putting it all together,

$$-\frac{3}{s}\mathrm{e}^{-st}\Big|_0^2 + \left(0 - \mathrm{e}^{-2s}\left(-\frac{4}{s} + \frac{1}{s^2}\right)\right) = -\frac{3\mathrm{e}^{-2s}}{s} + \frac{3}{s} + \frac{4\mathrm{e}^{-2s}}{s} - \frac{\mathrm{e}^{-2s}}{s^2} = \frac{3}{s} + \mathrm{e}^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

NOTE: Did you remember to *anti*differentiate in the third column?

(b)

$$f(t) = \begin{cases} \mathrm{e}^{-t}, & 0 \leq t < 5 \\ -1, & t \geq 5 \end{cases}$$

$$\int_0^\infty \mathrm{e}^{-st} f(t)\,dt = \int_0^5 \mathrm{e}^{-st}\mathrm{e}^{-t}\,dt + \int_5^\infty -\mathrm{e}^{-st}\,dt = \int_0^5 \mathrm{e}^{-(s+1)t}\,dt + \int_5^\infty -\mathrm{e}^{-st}\,dt$$

Taking the antiderivatives,

$$-\frac{1}{s+1}\mathrm{e}^{-(s+1)t}\Big|_0^5 + \frac{1}{s}\mathrm{e}^{-st}\Big|_5^\infty = \frac{1}{s+1} - \frac{\mathrm{e}^{-5(s+1)}}{s+1} + 0 - \frac{\mathrm{e}^{-5s}}{s}$$

3. Check your answers to Problem 2 by rewriting $f(t)$ using the step (or Heaviside) function, and use the table to compute the corresponding Laplace transform.

   (a) $f(t) = 3(u_0(t) - u_2(t)) + (6-t)u_2(t) = 3 - 3u_2(t) + (6-t)u_2(t) = 3 + (3-t)\,u_2(t)$
   For the second term, notice that the table entry is for $u_c(t)h(t-c)$. Therefore, if

$$h(t-2) = 3 - t \quad \text{then} \quad h(t) = 3 - (t+2) = 1 - t \quad \text{and} \quad H(s) = \frac{1}{s} - \frac{1}{s^2}$$

   Therefore, the overall transform is:

$$\frac{3}{s} + e^{-2s}\left(\frac{1}{s} - \frac{1}{s^2}\right)$$

   (b) $f(t) = e^{-t}\,(u_0(t) - u_5(t)) - u_5(t)$
   We can rewrite f in preparation for the transform:

$$f(t) = e^{-t}u_0(t) - e^{-t}u_5(t) - u_5(t)$$

   For the middle term,

$$h(t-5) = e^{-t} \quad \Rightarrow \quad h(t) = e^{-(t+5)} = e^{-5}e^{-t}$$

   so the overall transform is:

$$F(s) = \frac{1}{s+1} - e^{-5}\frac{e^{-5s}}{s+1} - \frac{e^{-5s}}{s}$$

4. Determine the Laplace transform:

   (a) t^2e^{-9t}

$$\frac{2}{(s+9)^3}$$

   (b) $e^{2t} - t^3 - \sin(5t)$

$$\frac{1}{s-2} - \frac{6}{s^4} - \frac{5}{s^2+25}$$

   (c) $t^2y'(t)$. Use Table Entry 16, $\mathcal{L}(-t^n f(t)) = F^{(n)}(s)$. In this case, $F(s) = sY(s) - y(0)$, so $F'(s) = sY'(s) + Y(s)$ and $F''(s) = sY''(s) + 2Y'(s)$.

   (d) $e^{3t}\sin(4t)$

$$\frac{4}{(s-3)^2+16}$$

   (e) $e^t\delta(t-3)$
   In this case, notice that $f(t)\delta(t-c)$ is the same as $f(c)\delta(t-c)$, since the delta function is zero everywhere except at $t = c$. Therefore,

$$\mathcal{L}(e^t\delta(t-c)) = e^3e^{-3s}$$

(f) $t^2 u_4(t)$

In this case, let $h(t-4) = t^2$, so that

$$h(t) = (t+4)^2 = t^2 + 8t + 16 \quad \Rightarrow \quad H(s) = \frac{2 + 8s + 16s^2}{s^3}$$

and the overall transform is $\mathrm{e}^{-4s} H(s)$.

5. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$

$$\frac{2s-1}{s^2-4s+6} = \frac{2s-1}{(s^2-4s+4)+2} = 2\frac{s-1/2}{(s-2)^2+2} =$$

In the numerator, make $s - \frac{1}{2}$ into $s - 2 + \frac{3}{2}$, then

$$2\left(\frac{s-2}{(s-2)^2+2} + \frac{3}{2\sqrt{2}}\frac{\sqrt{2}}{(s-2)^2+2}\right) \Rightarrow 2\mathrm{e}^{2t}\cos(\sqrt{2}t) + \frac{3}{\sqrt{2}}\mathrm{e}^{2t}\sin(\sqrt{2}t)$$

(b) $\dfrac{7}{(s+3)^3} = \dfrac{7}{2!}\dfrac{2!}{(s+3)^3} \Rightarrow \dfrac{7}{2}t^2\mathrm{e}^{-3t}$

(c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)} = \mathrm{e}^{-2s}H(s)$, where

$$H(s) = \frac{4s+2}{(s-1)(s+2)} = \frac{2}{s-1} + \frac{2}{s+2} \quad \Rightarrow \quad h(t) = 2\mathrm{e}^{t} + 2\mathrm{e}^{-2t}$$

and the overall inverse: $u_2(t)h(t-2)$.

(d) $\dfrac{3s-1}{2s^2-8s+14}$ Complete the square in the denominator, factoring the constants out:

$$\frac{3s-1}{2(s^2-8s+5)} = \frac{3}{2}\cdot\frac{s-1/3}{(s-2)^2+3} = \frac{3}{2}\left(\frac{s-2}{(s-2)^2+3} + \frac{5}{3}\cdot\frac{1}{\sqrt{3}}\frac{\sqrt{3}}{(s-2)^2+3}\right)$$

The inverse transform is:

$$\frac{3}{2}\mathrm{e}^{2t}\cos(\sqrt{3}t) + \frac{5}{2\sqrt{3}}\mathrm{e}^{2t}\sin(\sqrt{3}t)$$

(e) $\left(\mathrm{e}^{-2s} - \mathrm{e}^{-3s}\right)\dfrac{1}{s^2+s-6} = \left(\mathrm{e}^{-2s} - \mathrm{e}^{-3s}\right)H(s)$

Where:

$$H(s) = \frac{1}{s^2+s-6} = \frac{1}{5}\frac{1}{s-2} - \frac{1}{5}\frac{1}{s+3}$$

so that

$$h(t) = \frac{1}{5}\mathrm{e}^{2t} - \frac{1}{5}\mathrm{e}^{-3t}$$

and the overall transform is:

$$u_2(t)h(t-2) - u_3(t)h(t-3)$$

6. For the following differential equations, solve for $Y(s)$ (the Laplace transform of the solution, $y(t)$). Do not invert the transform.

   (a) $y'' + 2y' + 2y = t^2 + 4t$, $y(0) = 0$, $y'(0) = -1$

   $$s^2Y + 1 + 2sY + 2Y = \frac{2}{s^3} + \frac{4}{s^2}$$

   so that

   $$Y(s) = \frac{2}{s^3(s^2+2s+2)} + \frac{4}{s^2(s^2+2s+2)} - \frac{1}{s^2+2s+2}$$

   (b) $y'' + 9y = 10\mathrm{e}^{2t}$, $y(0) = -1$, $y'(0) = 5$

   $$s^2Y + s - 5 + 9Y = \frac{10}{s-2} \Rightarrow Y(s) = \frac{10}{(s-2)(s^2+9)} - \frac{s-5}{s^2+9}$$

   (c) $y'' - 4y' + 4y = t^2\mathrm{e}^t$, $y(0) = 0$, $y'(0) = 0$

   $$(s^2 - 4s + 4)Y = \frac{2}{(s-1)^3} \Rightarrow Y(s) = \frac{2}{(s-1)^3(s-2)^2}$$

7. Solve the given initial value problems using Laplace transforms:

   (a) $2y'' + y' + 2y = \delta(t-5)$, zero initial conditions.

   $$Y = \frac{\mathrm{e}^{-5s}}{2s^2+s+2} = \mathrm{e}^{-5s}H(s)$$

   where

   $$H(s) = \frac{1}{2s^2+s+2} = \frac{1}{2}\frac{1}{s^2+\frac{1}{2}s+1} = \frac{1}{2}\frac{1}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} = \frac{1}{2}\frac{4}{\sqrt{15}}\frac{\frac{\sqrt{15}}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$$

   Therefore,

   $$h(t) = \frac{2}{\sqrt{15}}\mathrm{e}^{-1/4\,t}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

   And the overall solution is $u_5(t)h(t-5)$

   (b) $y'' + 6y' + 9y = 0$, $y(0) = -3$, $y'(0) = 10$

   $$s^2Y + 3s - 10 + 6(sY+3) + 9Y = 0 \quad \Rightarrow \quad Y = -\frac{3s+8}{(s+3)^2}$$

   Partial Fractions:

   $$-\frac{3s+8}{(s+3)^2} = -\frac{3}{(s+3)} + \frac{1}{(s+3)^2} \Rightarrow y(t) = -3\mathrm{e}^{-3t} + t\mathrm{e}^{-3t}$$

(c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$

$$Y = e^{-s}\frac{1}{s(s-3)(s+1)} + \frac{1}{(s+1)(s-3)} = e^{-s}H(s) + \frac{1}{4}\frac{1}{s-3} - \frac{1}{4}\frac{1}{s+1}$$

where

$$H(s) = \frac{1}{s(s-3)(s+1)} = -\frac{1}{3}\frac{1}{s} + \frac{1}{12}\frac{1}{s-3} + \frac{1}{4}\frac{1}{s+1}$$

so that

$$h(t) = -\frac{1}{3} + \frac{1}{12}e^{3t} + \frac{1}{4}e^{-t}$$

and the overall solution is:

$$y(t) = \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} + u_1(t)h(t-1)$$

(d) $y'' + 4y = \delta(t - \frac{\pi}{2})$, $y(0) = 0$, $y'(0) = 1$

$$Y = e^{-\pi/2\,s}\frac{1}{s^2+4} + \frac{1}{s^2+4}$$

Therefore,

$$y(t) = \frac{1}{2}\sin(2t) + u_{\pi/2}(t)\frac{1}{2}\sin(2(t-\pi/2))$$

(e) $y'' + y = \sum_{k=1}^{\infty}\delta(t - 2k\pi)$, $y(0) = y'(0) = 0$. Write your answer in piecewise form.

$$Y(s) = \sum_{k=1}^{\infty} e^{-2k\pi s}\frac{1}{s^2+1}$$

Therefore, term-by-term,

$$y(t) = \sum_{k=1}^{\infty} u_{2k\pi}(t)\sin(t - 2\pi k) = \sum_{k=1}^{\infty} u_{2\pi k}(t)\sin(t)$$

Piecewise,

$$y(t) = \begin{cases} 0 & \text{if} \quad 0 \le t < 2\pi \\ \sin(t) & \text{if} \quad 2\pi \le t < 4\pi \\ 2\sin(t) & \text{if} \quad 4\pi \le t < 6\pi \\ 3\sin(t) & \text{if} \quad 6\pi \le t < 8\pi \\ \vdots & \vdots \end{cases}$$

8. Short Answer:

(a) $\int_0^{\infty} \sin(3t)\delta(t - \frac{\pi}{2})\,dt = \sin(3\pi/2) = -1$, since

$$\int_0^{\infty} f(t)\delta(t-c)\,dt = f(c)$$

(b) If $y'' + 2y' + 3y = 0$ and $y(0) = 1$, $y'(0) = -1$, compute $y''(0)$, $y'''(0)$, and $y^{(4)}(0)$.

We see that:

$$y'' = -2y' - 3y \text{ at } x = 0 \Rightarrow y''(0) = -2(-1) - 3(1) = -1$$

$$y''' = -2y'' - 3y' \text{ at } x = 0 \Rightarrow y'''(0) = (-2)(-1) - 3(-1) = 5$$

$$y^{(4)} = -2y''' - 3y'' \text{ at } x = 0 \Rightarrow y^{(4)}(0) = (-2)(5) - 3(-1) = -7$$

(c) Using your previous result, give the Taylor expansion of the solution to the differential equation using at least 5 terms.

$$y(x) = 1 - x - \frac{1}{2!}x^2 + \frac{5}{3!}x^3 - \frac{7}{4!}x^4 + \ldots$$

(d) If $y'(t) = \delta(t - c)$, what is $y(t)$?

We could solve formally using Laplace transforms:

$$sY - y(0) = \mathrm{e}^{-cs} \Rightarrow Y = \frac{\mathrm{e}^{-cs}}{s} + \frac{y(0)}{s}$$

so that $y(t) = u_c(t) + y(0)$, where $y(0)$ we can take to be an arbitrary constant.

(e) What is the expected radius of convergence for the series expansion of $f(x) = 1/(x^2 + 2x + 5)$ if the series is based at $x_0 = 1$?

Not on the exam.

(f) Use Laplace transforms to solve for $F(s)$, if

$$f(t) + 2\int_0^t \cos(t - x) f(x)\, dx = \mathrm{e}^{-t}$$

(So only solve for the transform of $f(t)$, don't invert it back).

$$F(s) + 2F(s)\frac{s}{s^2 + 1} = \frac{1}{s + 1} \quad \Rightarrow \quad F(s)\left(\frac{(s + 1)^2}{s^2 + 1}\right) = \frac{1}{s + 1}$$

so that

$$F(s) = \frac{s^2 + 1}{(s + 1)^3}$$

(g) In order for the Laplace transform of f to exist, f must be?

f must be piecewise continuous and of exponential order

(h) Can we assume that the solution to: $y'' + p(x)y' + q(x)y = u_3(x)$ is a power series?

No. Notice that the second derivative is not continuous at $x = 3$, but the second derivative of the power series would be.

(i) Is $x = 0$ an ordinary points of $y'' + \sqrt{x}y' + xy = x^2$?

NOTE: The definition of an ordinary point changes between Section 5.2 and Section 5.3. See pages 250 and 262 of the text (because we did not get into 5.3, this will not be on the exam).

9. More on Laplace Transforms:

   (a) Your friend tells you that the solutions to the IVPs:

   $$y''+2y'+y=0,\quad y(0)=0,\ y'(0)=1 \qquad \text{and} \qquad y''+2y'+y=\delta(t)\quad y(0)=0,\ y'(0)=0$$

   are exactly the same. Are they really? Explain.

   Both models give the same solution if $t \geq 0$. If we consider all time, then the solutions are different.

   Conceptually, the two IVPs are also modeling different behavior. In the second IVP, we are modeling a "hit" at time zero, but in the first, the spring-mass system (for example), is simply going through equilibrium at a velocity of 1.

   By the way, the solution to both IVPs is

   $$y(t) = te^{-t}$$

   Valid for all positive time in both models, valid for all time in the first, only valid for $t \geq 0$ in the second (in the Dirac model, the function would be zero for all negative time due to the initial conditions).

   (b) Let $f(t) = t$ and $g(t) = u_2(t)$.

      i. Use the Laplace transform to compute $f * g$.
      To use the table,

      $$\mathcal{L}(t * u_2(t)) = \frac{1}{s^2} \cdot \frac{e^{-2s}}{s} = e^{-2s}\frac{1}{s^3} = e^{-2s}H(s)$$

      so that $h(t) = \frac{1}{2}t^2$. The inverse transform is then

      $$u_2(t)\frac{1}{2}(t-2)^2$$

      ii. Verify your answer by directly computing the integral.
      By direct computation, we'll choose to "flip and shift" the function t:

      $$f * g = \int_0^t (t-x)u_2(x)\,dx$$

      Notice that $u_2(x)$ is zero until $x = 2$, then $u_2(x) = 1$. Therefore, if $t \leq 2$, the integral is zero. If $t \geq 2$, then:

      $$\int_0^t (t-x)u_2(x)\,dx = \int_2^t t - x\,dx = tx - \frac{1}{2}x^2\Big|_2^t = t^2 - \frac{1}{2}t^2 - 2t + 2 = \frac{1}{2}(t-2)^2$$

      valid for $t \geq 2$, zero before that. This means that the convolution is:

      $$t * u_2(t) = \frac{1}{2}(t-2)^2 u_2(t)$$

10. Find the recurrence relation between the coefficients for the power series solutions to the following:

(a) $2y'' + xy' + 3y = 0$, $x_0 = 0$.

Substituting our power series in for y, y', y'':

$$2\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} + 3\sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} 3a_n x^n = 0$$

Noting that in the second sum we could start at $n = 0$, since the first term (constant term) would be zero anyway, we can start all series with a constant term:

$$\sum_{k=0}^{\infty} \left(2(k+2)(k+1)a_{k+2} + ka_k + 3a_k\right) x^k = 0$$

From which we get the recurrence relation:

$$a_{k+2} = -\frac{k+3}{2(k+2)(k+1)}\, a_k$$

(b) $(1-x)y'' + xy' - y = 0$, $x_0 = 0$

Substituting our power series in for y, y', y'':

$$(1-x)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + x\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

We want to write this as a single sum, with each index starting at the same value. First we'll simplify a bit:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

The two middle sums can have their respective index taken down by one (so that formally the series would start with $0x^0$):

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now make all the indices the same. To do this, in the first sum make $k = n-2$, in the second sum take $k = n-1$. Doing this and collecting terms:

$$\sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - (k+1)ka_{k+1} + (k-1)a_k\right) x^k = 0$$

So we get the recursion:

$$a_{k+2} = \frac{(k+1)k\, a_{k+1} - (k-1)a_k}{(k+2)(k+1)}$$

(c) $y'' - xy' - y = 0$, $x_0 = 1$

Done in class;

$$a_{n+2} = \frac{1}{n+2}\left(a_{n+1} + a_n\right)$$

11. Find the first 5 terms of the power series solution to $e^x y'' + xy = 0$ if $y(0) = 1$ and $y'(0) = -1$.

Compute the derivatives directly, then (don't forget to divide by $n!$):

$$y(x) = 1 - x - \frac{1}{3!}x^3 + \frac{1}{3!}x^4 + \dots$$

12. Find the radius of convergence for the following series:

(a) $\sum_{n=1}^{\infty} \sqrt{n}x^n \quad \rho = 1$

(b) $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n+1}}(x+3)^n \quad \rho = 1/2$

(c) $\sum_{n=1}^{\infty} \frac{n!\,x^n}{n^n} \quad \rho = e$ (From the HW)