M244: Review Solutions

1. Solve (use any method if not otherwise specified):

   (a) $(2x - 3x^2)\frac{dx}{dt} = t\cos(t)$ This is separable, and separated.

   $$\int 2x - 3x^2\,dx = \int t\cos(t)\,dt \Rightarrow x^2 - x^3 = \cos(t) + t\sin(t) + C$$

   You can leave your answer in implicit form.

   (b) $y'' + 2y' + y = \sin(3t)$ Get the homog part then the particular solution (or use Laplace):

   $$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1, -1 \Rightarrow y_h(t) = \mathrm{e}^{-t}\left(C_1 + C_2 t\right)$$

   For the particular solution, (Undet Coefs), $y_p = A\cos(3t) + B\sin(3t)$. Substitute to get:

   $$(A - 6B - 9A)\sin(3t) + (B + 6A - 9B)\cos(3t) = \sin(3t)$$

   so that $A = -2/25$, $B = -3/50$. Altogether,

   $$y(t) = \mathrm{e}^{-t}\left(C_1 + C_2 t\right) - \frac{2}{25}\sin(3t) - \frac{3}{50}\cos(3t)$$

   (c) $y'' - 3y' + 2y = \mathrm{e}^{2t}$ Same technique here. The roots to the characteristic equation are $r = 1, 2$, so the homogeneous part of the solution is:

   $$y_h(t) = C_1\mathrm{e}^{t} + C_2\mathrm{e}^{2t}$$

   Initially, we guess that $y_p(t) = A\mathrm{e}^{2t}$, but that is part of y_h, so multiply by t: $y_p = At\mathrm{e}^{2t}$. Now substitute into the D.E. to get: $A = 1$. The full solution is

   $$y(t) = C_1\mathrm{e}^{t} + C_2\mathrm{e}^{2t} + t\mathrm{e}^{2t}$$

   (d) $x' = \sqrt{t}\mathrm{e}^{-t} - x$. This is a linear differential equation, with integrating factor; $x' + x = \sqrt{t}\mathrm{e}^{-t}$. The integrating factor is $\mathrm{e}^{\int 1\,dt} = \mathrm{e}^{t}$. Therefore,

   $$(x\mathrm{e}^{t})' = \sqrt{t} \Rightarrow x\mathrm{e}^{t} = \frac{2}{3}t^{3/2} + C \Rightarrow x = \left(\frac{2}{3}t^{3/2} + C\right)\mathrm{e}^{-t}$$

   (e) $x' = 2 + 2t^2 + x + t^2 x$. This is a linear differential equation: $x' - (1+t^2)x = 2(1+t^2)$. The integrating factor is:

   $$\mathrm{e}^{\int -(1+t^2)\,dt} = \mathrm{e}^{-t-(1/3)t^3}$$

   So we solve the following (to integrate, let $u = t + (1/3)t^2$)

   $$\left(x\mathrm{e}^{-t-(1/3)t^3}\right)' = 2(1+t^2)\mathrm{e}^{-t-(1/3)t^3} \Rightarrow \left(x\mathrm{e}^{-t-(1/3)t^3}\right) =$$

   $$-2\mathrm{e}^{-t-(1/3)t^3} + C \Rightarrow x = -2 + C\mathrm{e}^{t+(1/3)t^3}$$

2. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:

$$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

The general solution (before initial conditions):

$$y(t) = C_1 \mathrm{e}^{3t} + C_2 \mathrm{e}^{-2t}$$

With the initial conditions,

$$1 = C_1 + C_2 \quad \alpha = 3C_1 - 2C_2 \Rightarrow C_1 = \frac{2+\alpha}{5}, \quad C_2 = \frac{3-\alpha}{5}$$

Therefore,

$$y(t) = \frac{2+\alpha}{5}\mathrm{e}^{3t} + \frac{3-\alpha}{5}\mathrm{e}^{-2t}$$

For $y(t) \to 0$, we must have $\alpha = -2$ (to zero out the first term).

3. The Wronskian of two functions is $W(t) = t^2 - 4$.

Can they form a fundamental set of solutions to a second order linear differential equation? No, unless the interval for the solution does not contain $t = \pm 2$. For example, if $t \in (0, 2)$, then the Wronskian could be $t^2 - 4$. (This is a consequence of Abel's Theorem)

4. Compute $\mathcal{L}(\cos(t))$

The idea to use complex exponentials comes from Euler's Formula:

$$\int_0^\infty \mathrm{e}^{bit}\mathrm{e}^{-st}\,dt = \int_0^\infty \mathrm{e}^{-st}\cos(bt)\,dt + i\int_0^\infty \mathrm{e}^{-st}\sin(bt)\,dt$$

Therefore, the right side of the equation will be easy to compute, and the formula above tells us to take the real part of the expression:

$$\int_0^\infty \mathrm{e}^{it}\mathrm{e}^{-st}\,dt = \int_0^\infty \mathrm{e}^{-(s-i)t}\,dt = -\frac{1}{s-i}\mathrm{e}^{-(s-i)t}\Big|_0^\infty$$

The constant in the front does not contribute to the limit as $t \to \infty$, so the limit is determined by:

$$\lim_{t\to\infty} \mathrm{e}^{-(s-i)t} = \lim_{t\to\infty} \mathrm{e}^{-st}(\cos(t) + i\sin(t))$$

This limit is zero as long as $s > 0$ (the complex part is always on the unit circle).

Therefore, the integral converges to

$$\frac{1}{s-i} = \frac{1}{s-i}\cdot\frac{s+i}{s+i} = \frac{s+i}{s^2+1} = \frac{s}{s^2+1} + i\frac{1}{s^2+1}$$

The Laplace transform of $\cos(t)$ is the first expression (and we get the Laplace transform of the sine as well!).

Alternate Solution without complex exponentials:

$$\int_0^\infty \mathrm{e}^{-st}\cos(t)\,dt = \mathrm{e}^{-st}\left(\sin(t) - s\,\cos(t)\right) - s^2\int_0^\infty \mathrm{e}^{-st}\cos(t)\,dt$$

so that

$$(1+s^2)\int_0^\infty \mathrm{e}^{-st}\cos(t)\,dt = \mathrm{e}^{-st}\left(\sin(t) - s\,\cos(t)\right)$$

and:

$$\int_0^\infty \mathrm{e}^{-st}\cos(t)\,dt = \frac{\mathrm{e}^{-st}}{s^2+1}\left(\sin(t) - s\,\cos(t)\right)\Big|_0^\infty = \frac{1}{s^2+1}\left(\lim_{T\to\infty}\frac{\sin(T)-s\cos(T)}{\mathrm{e}^{sT}} + s\right)$$

To compute the limit, note that:

$$\frac{-1}{\mathrm{e}^{sT}} \le \frac{\sin(t)}{\mathrm{e}^{sT}} \le \frac{1}{\mathrm{e}^{sT}}$$

so that, by the Squeeze Theorem, the overall limit is zero (same if the numerator were $\cos(t)$).

Put everything together now to say that:

$$\int_0^\infty \mathrm{e}^{-st}\cos(t)\,dt = \frac{1}{s^2+1}\cdot(0+s) = \frac{s}{s^2+1}$$

5. Given the equation $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$, we need to divide to put it in the right form for the E & U theorem:

$$y'' - \frac{2x}{1-x^2}y' + \frac{\alpha(\alpha+1)}{1-x^2}y = 0$$

The E & U Theorem will guarantee that a unique solution exists as long as $x_0 \neq \pm 1$. In these cases, the interval on which the solution exists will be either $(-\infty,-1)$, $(-1,1)$ or $(1,\infty)$.

The Wronskian of two solutions is given by Abel's Theorem:

$$W(y_1,y_2) = C\mathrm{e}^{-\int p(x)\,dx}$$

In this case,

$$-\int p(x)\,dx = \int \frac{2x}{1-x^2}\,dx = -\ln|1-x^2|$$

so that the Wronskian is:

$$\frac{C}{1-x^2}$$

6. Let $y''' - y' = t\mathrm{e}^{-t} + 2\cos(t)$. First, use our ansatz to find the characteristic equation for the third order homogeneous equation. Determine a suitable form for the particular solution, y_p using Undetermined Coefficients. Do not solve for the coeffs.

The ansatz was $y = \mathrm{e}^{rt}$, so that $y' = r\mathrm{e}^{rt}$ and $y''' = r^3\mathrm{e}^{rt}$. Therefore, the homogeneous equation becomes:

$$y''' - y' = 0 \Rightarrow \mathrm{e}^{rt}\left(r^3 - r\right) = 0$$

so that $r(r^2-1) = 0$. Therefore, $r = 0$, $r = \pm 1$. Extrapolating from the second order differential equation, we expect the homogeneous solution to be:

$$y_h = C_1 + C_2\mathrm{e}^t + C_3\mathrm{e}^{-t}$$

and the form for the particular solution (break into two pieces):

$$y_{p_1} = (At + B)\mathrm{e}^{-t} \Rightarrow y_{p_1} = t(At + B)\mathrm{e}^{-t}$$

and

$$y_{p_2} = A\cos(t) + B\sin(t)$$

7. A tank contains 200 gallons of water with 100 pounds of salt. Water containing 1 pound of salt per gallon is entering at a rate of 3 gallons per minute. The well-mixed solution is pumped out at a rate of 2 gallons per minute. Find the concentration of salt in the tank at time t (assuming the tank can hold it).

   The differential equation is:

$$Q' = 3 - \frac{2}{200 + t}Q \qquad Q(0) = 100$$

   This is linear:

$$Q' + \frac{2}{200 + t}Q = 3$$

   and the integrating factor is: $(200 + t)^2$ so that:

$$\left(Q(200 + t)^2\right) = (200 + t)^3 + C \quad \Rightarrow \quad Q = 200 + t + \frac{C}{(200 + t)^2}$$

   Using the initial condition, we see that:

$$Q(t) = 200 + t - \frac{4,000,000}{(200 + t)^2}$$

8. Suppose that we have a mass-spring system modelled by the differential equation

$$x'' + 2x' + x = 0, x(0) = 2, x'(0) = -3$$

   Find the solution, and determine whether the mass ever crosses $x = 0$. If it does, determine the velocity at that instant. See if it crosses if the initial velocity is cut in half.

   The solution is: $x(t) = \mathrm{e}^{-t}(2 - t)$, which crosses $x = 0$ when $t = 2$. If we half the initial velocity, the solution is: $x(t) = \mathrm{e}^{-t}\left(2 + \frac{1}{2}t\right)$, which does not cross $x = 0$ in positive time.

9. Let $y(x)$ be a power series solution to $(1 - x)y'' + y = 0,\ x_0 = 0$. Find the recurrence relation, and write the first 5 terms of the expansion of y.

   Let $y = \sum_{n=0}^{\infty} c_n x^n$. Substitute into the differential equation:

$$(1 - x)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

We need to get the same index for each sum. Convert everything to start with $k = 0$. Notice that the middle sum could start with $n = 1$ instead of $n = 2$ and it would have no effect.

That means that in the first sum, let $k = n - 2$. In the second sum, let $k = n - 1$, and in the last sum, $k = n$. Then:

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=0}^{\infty}(k+1)k\, c_{k+1}x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

Put it all together:

$$\sum_{k=0}^{\infty}\left((k+2)(k+1)c_{k+2} - (k+1)kc_{k+1} + c_k\right)x^k = 0$$

And we get the recurrence relation:

$$c_{k+2} = \frac{k}{k+2}c_{k+1} - \frac{1}{(k+1)(k+2)}c_k$$

From this, we can write everything in terms of C_0, C_1:

$$y(x) = C_0 + C_1 x - \frac{1}{2}C_0 x^2 - \frac{1}{6}\left(C_0 + C_1\right)x^3 - \frac{1}{24}\left(C_0 + 2C_1\right)x^4 + \ldots$$

(Note: On the exam, I would give you initial conditions so you wouldn't have to compute these symbolically in terms of C_0, C_1)

10. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$. Find the recurrence relation and write the first 5 terms of the expansion of y.

Same idea as the previous exercise, but be careful to use powers of $(x - 1)$!

$$\sum_{n=2}^{\infty} n(n-1)c_{n-2}(x-1)^{n-2} - x\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} - \sum_{n=0}^{\infty} c_n(x-1)^n = 0$$

For the middle sum, recall our trick: $x = 1 + (x-1)$, which gives us a way of incorporating the x into the sum:

$$x\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} = (1+(x-1))\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} = \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} + \sum_{n=1}^{\infty} nc_n(x-1)^n$$

Now shift the index of every sum to match, and you should get the recurrence relation:

$$C_{k+2} = \frac{1}{k+2}\left(C_{k+1} + C_k\right)$$

We get:

$$y(x) = C_0 + C_1(x-1) + \frac{C_0 + C_1}{2}(x-1)^2 + \frac{C_0 + 3C_1}{6}(x-1)^3 + \frac{2C_0 + 3C_1}{12}(x-1)^4 + \ldots$$

We can check our answer using the next exercise...

11. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$ (the same as the previous DE), with $y(1) = 1$ and $y'(1) = 2$. Compute the first 5 terms of the power series solution by first computing $y''(1), y'''(1), y^{(4)}(1)$.

First, let's compute the derivatives:

$$y'' = xy' + y \qquad \text{at } x = 1 \quad \Rightarrow y''(1) = y'(1) + y(1)$$

so that:

$$y''' = xy'' + 2y' \qquad \text{at } x = 1 \quad \Rightarrow y'''(1) = y''(1) + 2y'(1)$$

and:

$$y^{(4)} = xy''' + 3y'' \qquad \text{at } x = 1 \quad \Rightarrow y^{(4)}(1) = y'''(1) + 3y''(1)$$

From this, we see that:

$$y''(1) = 3, \qquad y'''(1) = 7, \qquad y^{(4)}(1) = 16$$

Writing out the solution:

$$y(x) = 1 + 2(x-1) + \frac{3}{2}(x-1)^2 + \frac{7}{6}(x-1)^3 + \frac{16}{24}(x-1)^4 + \ldots$$

12. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

$$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6 - t, & 2 < t \end{cases}$$

$$\int_0^\infty f(t)\mathrm{e}^{-st}\,dt = 3\int_0^2 \mathrm{e}^{-st}\,dt + \int_2^\infty (6-t)\mathrm{e}^{-st}\,dt =$$
$$\frac{3}{s}(1 - \mathrm{e}^{-2s}) + \frac{\mathrm{e}^{-2s}}{s^2}(4s - 1)$$

13. Determine the Laplace transform:

(a) $t^2\mathrm{e}^{-9t} \Rightarrow \dfrac{2}{(s+9)^3}$

(b) $\mathrm{e}^{2t} - t^3 - \sin(5t) \Rightarrow \dfrac{1}{s-2} - \dfrac{6}{s^4} - \dfrac{5}{s^2+25}$

(c) $u_5(t)(t-5)^4 \Rightarrow \dfrac{24\mathrm{e}^{-5s}}{s^5}$

(d) $\mathrm{e}^{3t}\sin(4t) \Rightarrow \dfrac{4}{(s-3)^2 + 16}$

(e) $\mathrm{e}^t\delta(t-3) \Rightarrow \mathrm{e}^{-3s+3}$

(f) $t^2u_4(t) \Rightarrow \mathrm{e}^{-4s}\left(\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right)$

Note: Let $f(t-4) = t^2$, so that $f(t) = (t+4)^2 = t^2 + 8t + 16$, and $F(s) = \frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}$.

14. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$. Rewrite: $2\cdot\dfrac{s-2}{(s-2)^2+2}+\dfrac{3}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{(s-2)^2+2}$ The inverse is then $e^{2t}\left(2\cos(\sqrt{2}t+\dfrac{3}{\sqrt{2}}\sin(\sqrt{2}t)\right)$

(b) $\dfrac{7}{(s+3)^3}\Rightarrow\dfrac{7}{2}t^2e^{-3t}$

(c) $\dfrac{e^{-2s}(4s+2)}{(s-1)(s+2)}$. You might rewrite this as $e^{-2s}H(s)$, where

$$H(s)=\frac{4s+2}{(s-1)(s+2)}=\frac{2}{s+2}+\frac{2}{s-1}$$

Now, $h(t)=2e^{-2t}+2e^t$, and the solution is $u_2(t)h(t-2)$.

(d) $\dfrac{3s-2}{(s-4)^2-3}$ We might rewrite this as:

$$3\cdot\frac{s-4}{(s-4)^2-3}+\frac{10}{\sqrt{3}}\cdot\frac{\sqrt{3}}{(s-4)^2-3}=3F(s-4)+\frac{10}{\sqrt{3}}G(s-4)$$

where $F(s)=\dfrac{s}{s^2-3}$, $G(s)=\dfrac{\sqrt{3}}{s^2-3}$. The inverse is (Item 14 from the Table):

$$e^{4t}\left(3f(t)+\frac{10}{\sqrt{3}}g(t)\right)=e^{4t}\left(3\cosh(\sqrt{3}t)+\frac{10}{\sqrt{3}}\sinh(\sqrt{3}t)\right)$$

15. Solve the given initial value problems using Laplace transforms:

(a) $y''+2y'+2y=4t$, $y(0)=0$, $y'(0)=-1$. The Laplace transform:

$$Y(s)=\frac{4-s^2}{s^2(s^2+2s+2)}=-\frac{2}{s}+\frac{2}{s^2}+\frac{2s+1}{s^2+2s+2}=$$

$$-\frac{2}{s}+\frac{2}{s^2}+2\frac{s+1}{(s+1)^2+1}-\frac{1}{(s+1)^2+1}$$

so that

$$y(t)=-2+2t+e^{-t}\left(2\cos(t)-\sin(t)\right)$$

(b) $y''+9y=10e^{2t}$, $y(0)=-1$, $y'(0)=5$. Expanding, we get

$$Y(s)=\frac{10}{(s-2)(s^2+9)}-\frac{s-5}{s^2+9}=\frac{10}{13}\cdot\frac{1}{s-2}-\frac{23}{13}\cdot\frac{s}{s^2+9}+\frac{15}{13}\cdot\frac{3}{s^2+9}$$

so that

$$y(t)=\frac{10}{13}e^{2t}-\frac{23}{13}\cos(3t)+\frac{15}{13}\sin(3t)$$

(c) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$ Use partial fractions:

$$Y(s) = -\frac{1}{4}\cdot\frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{s-3} + \mathrm{e}^{-s}\left(-\frac{1}{3}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s+1} + \frac{1}{12}\cdot\frac{1}{s-3}\right)$$

Think of this second term as $\mathrm{e}^{-s}\cdot H(S)$, where

$$h(t) = -\frac{1}{3} + \frac{1}{4}\mathrm{e}^{-t} + \frac{1}{12}\mathrm{e}^{3t}$$

and the solution is:

$$y(t) = -\frac{1}{4}\mathrm{e}^{-t} + \frac{1}{4}\mathrm{e}^{3t} + u_1(t)h(t-1)$$

(d) $y'' - 4y' + 4y = t^2\mathrm{e}^t$, $y(0) = 0$, $y'(0) = 0$.

$$Y(s) = \frac{2}{(s-1)^3(s-2)^2} = 2\cdot\frac{2}{(s-1)^3}\cdot\frac{1}{(s-2)^2} = F(s)G(S)$$

Where $f(t) = t^2\mathrm{e}^t$ and $g(t) = t\mathrm{e}^{2t}$. Therefore,

$$y(t) = \int_0^t (t-\tau)^2\mathrm{e}^{t-\tau}\cdot\tau\mathrm{e}^{2\tau}\,d\tau$$

16. Evaluate: $\int_0^\infty \sin(3t)\delta(t-\frac{\pi}{2})\,dt = \sin(3\pi/2) = -1$

17. If $y'(t) = \delta(t-c)$, what is $y(t)$? Using Laplace,

$$sY - y(0) = \mathrm{e}^{-cs} \Rightarrow Y = \mathrm{e}^{-cs}\cdot\frac{1}{s} + y(0)\cdot\frac{1}{s} \Rightarrow y(t) = u_c(t) + y(0) \quad \text{or simply } y(t) = u_c(t)$$

18. What was the *ansatz* we used to obtain the characteristic equation for $ay'' + by' + cy = 0$? for $\mathbf{x}' = A\mathbf{x}$?

Respectively, e^{rt} and $\mathrm{e}^{\lambda t}\mathbf{v}$.

19. **Typo:** The question should read: Given that $y_t = t^2$, find y_2 so that y_1, y_2 form a fundamental set of solutions (we did not define linear independence) to the DE:

$$t^2y'' - 4ty' + 6y = 0$$

(HINT: The Wronskian can be computed two ways).

SOLUTION: $W(y_1, y_2) = y_1y_2' - y_1'y_2 = t^2y_2' - 2ty_2$. And, by Abel's Theorem,

$$W(y_1, y_2) = C\mathrm{e}^{-\int p(t)\,dt}$$

In this case, $p(t) = -4/t$, so the Wronskian is of the form Ct^4. Therefore, y_2 must satisfy the first order linear ODE:

$$t^2y_2' - 2ty_2 = Ct^4 \quad\Rightarrow\quad y_2' - \frac{2}{t}y_2 = Ct^2 \quad\Rightarrow\quad \left(\frac{y_2}{t^2}\right)' = C$$

Therefore, $y_2(t) = Ct^3 + C_2t^2$. Since t^2 is already part of the solution, we only require that $y_2(t) = t^3$.

20. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

(a) $y' - 0.5y = \mathrm{e}^{2t} \qquad y(0) = 1$. This is a linear (integrating factor) differential equation; the solution will be valid for all time t.

$$y(t) = \frac{2}{3}\mathrm{e}^{2t} + \frac{1}{3}\mathrm{e}^{\frac{1}{2}t}$$

(b) $y'' + 4y' + 5y = 0, \quad y(0) = 1, y'(0) = 0$ This is a linear second order with constant coefficients; the solution will be valid for all time t.

$$y(t) = \mathrm{e}^{-2t}\left(2\sin(t) + \cos(t)\right)$$

(c) $y' = 1 + y^2$. This is a separable differential equation. with $f(y) = 1 + y^2$. The functions f f_y are continuous for all t and y, so existence and uniqueness applies. We have to solve the d.e. to find the interval:

$$\int \frac{1}{1+y^2}\,dy = t + C \Rightarrow \tan^{-1}(y) = t + C \Rightarrow y = \tan(t+C)$$

The solution is only valid for $\frac{-\pi}{2} \leq t + C \leq \frac{\pi}{2}$.

(d) $y' = \frac{1}{2}y(3-y)$. Similar to the previous problem (this is separable)

$$\int \frac{1}{3}\cdot\frac{1}{y} + \frac{1}{3}\cdot\frac{1}{3-y}\,dy = \frac{1}{2}t + C \Rightarrow \ln|y| - \ln|3-y| = \frac{3}{2}t + C_2$$

Now solve for y:

$$\frac{y}{3-y} = A\mathrm{e}^{3/2t} \Rightarrow y = \frac{3}{1 + B\mathrm{e}^{-3/2t}}$$

If the initial condition is positive, this is valid for all time (Draw the phase diagram and direction field for this autonomous DE to see why). If the initial condition is negative, we would need to find where (in positive time) the solution has a vertical asymptote. NOTE: We are assuming that $t \geq 0$.

(e) $\sin(2x)\mathrm{dx} + \cos(3y)\mathrm{dy} = 0$. You can treat this as a separable differential equation. We get:

$$\sin(3y) = \frac{3}{2}\cos(2x) + C$$

If we were to solve for y, we would see that the expression

$$\frac{3}{2}\cos(2x) + C$$

must be between -1 and 1 (which is the domain of the inverse sine). Given a specific value of C, we could do this, but it would be a bit messy and take some time.

(f) $y'' + 2y' + y = 2\mathrm{e}^{-t}, \quad y(0) = 0, y'(0) = 1$

$$y(t) = \mathrm{e}^{-t}\left(t + t^2\right)$$

This solution is valid for all t.

(g) $y' = xy^2$

$$\int y^{-2}\,dy = \frac{1}{2}x^2 + C \Rightarrow -\frac{1}{y} = \frac{x^2+C_2}{2} \Rightarrow y = \frac{-2}{x^2+C_2}$$

The interval for which the solution will be valid will depend on if $C_2 > 0$ (the solution will be valid for all x), or if $C_2 < 0$ (there will be a vertical asymptote where $x = \pm\sqrt{-C_2}$)

(h) $2xy^2 + 2y + (2x^2y + 2x)y' = 0$ This is an exact equation:

$$\frac{\partial}{\partial y}(2xy^2 + 2y) = 4xy + 2 = \frac{\partial}{\partial x}(2x^2y + 2x)$$

Recall that the solution will be (implicit) $F(x, y) = C$, where

$$F_x = 2xy^2 + 2y \Rightarrow F(x, y) = x^2y^2 + 2xy + h(x)$$

and

$$F_y = 2x^2y + 2x \Rightarrow F(x, y) = x^2y^2 + 2xy + g(y)$$

Comparing, we see $F(x, y) = x^2y^2 + 2xy$, and the implicit solution is:

$$x^2y^2 + 2xy = C$$

Here we will not be able to give an interval on which the solution is valid unless we isolate y, although we would have a requirement that $2x^2y + 2x \neq 0$, so that y' would be defined.

(i) $y'' + 4y = t^2 + 3e^t, y(0) = 0, y'(0) = 1$.

$$y(t) = \frac{1}{5}\sin(2t) - \frac{19}{40}\cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$$

The solution is valid for all t.

21. Suppose $y' = -ky(y-1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

Your graph should be an upside down parabola (vertex up). There are equilibrium solutions at $y = 0$ (unstable) and $y = 1$ (stable).

The solution is found by partial fractions:

$$\int \frac{1}{y(y-1)}\,dy = -kx + C \quad \Rightarrow \quad -\ln(y) + \ln(y-1) = -kx + C \quad \Rightarrow$$

$$\ln\left(\frac{y-1}{y}\right) = -kx + C \quad \Rightarrow \quad \frac{y-1}{y} = Ae^{-kx} \quad \Rightarrow \quad y = \frac{1}{1 - Ae^{-kx}}$$

22. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

This is separable: $y' = y^2(2+x) \Rightarrow y^{-2}\,dy = (2+x)\,dx$, so

$$y(x) = \frac{-2}{x^2 + 4x - 2}$$

This has vertical asymptotes at $x = -2 \pm \sqrt{6}$, so that the solution is valid only when $-2 - \sqrt{6} < x < -2 + \sqrt{6}$, or when x is approximately between -4.45 and 0.45. Between these vertical asymptotes, y has a minimum where its derivative is 0,

$$y' = y^2(2+x) = 0 \Rightarrow y = 0 \text{ or } x = -2$$

From our solution, we see that $y \neq 0$, so the minimum occurs at $x = -2$, and the minimum is: $-1/5$.

23. If $y(t)$ is a population at time t, what is the model for "exponential growth"? What is the model for growth with a "carrying capacity" in the environment (the logistic equation)?

Exponential Growth: $y' = ky$

Carrying Capacity (Logistic Equation): $y' = y(k_1 - k_2 y)$

24. We have two tanks, A and B with 20 and 30 gallons of fluid, respectively. Water is being pumped into Tank A at a rate of 2 gallons per minute, 2 ounces of salt per gallon. The well-mixed solution is pumped out of Tank A and into Tank B at a rate of 4 gallons per minute. Solution from Tank B is entering Tank A at a rate of 2 gallons per minute. Water is being pumped into Tank B at k gallons per minute with 3 ounces of salt per gallon. The solution is being pumped out of tank B at a total rate of 5 gallons per minute (2 of them are going into tank A).

- What should k be in order for the amount of solution in Tank B to remain at 30? Use this value for the remaining problems.

  $k = 1$

- Write the system of differential equations for the amount of salt in Tanks A, B at time t. Do not solve.

  Let $A(t), B(t)$ be the amount (in ounces) of salt in Tanks A, B respectively. Then:

$$\frac{dA}{dt} = 4 + \frac{2}{30}B - \frac{4}{20}A$$

$$\frac{dB}{dt} = 3 + \frac{4}{20}A - \frac{5}{30}B$$

- Find the equilibrium solution and classify it.

  Solve:

$$\begin{aligned} 4 - \tfrac{1}{5}A + \tfrac{1}{15}B &= 0 \\ 3 + \tfrac{1}{5}A - \tfrac{1}{6}B &= 0 \end{aligned} \quad \Rightarrow \quad A = \frac{130}{3} \quad B = 70$$

  Notice that these quantities are in total ounces in each tank.

We can classify this equilibrium by looking at the matrix associated with the system:

$$\begin{bmatrix} -\frac{1}{5} & \frac{1}{15} \\ \frac{1}{5} & -\frac{1}{6} \end{bmatrix}$$

The trace is negative, the determinant is positive, and the discriminant is positive. The equilibrium is a SINK (which is what we would expect from the physical problem).

25. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

$$y(t) = \frac{-y_0}{y_0 t^2 - 1}$$

If $y_0 > 0$, then the solution will only be valid between $\pm\frac{1}{\sqrt{y_0}}$. If $y_0 < 0$, the solution will be valid for all t.

26. (Note: Think of this as a simplified set of DEs that we might have for two tanks) Given the system:

$$\begin{aligned} x' &= 3x - y - 2 \\ y' &= 4x - 2y \end{aligned}$$

(a) For the equilibria, set the derivatives to zero and solve:

$$\begin{aligned} 3x - y &= 2 \\ 4x - 2y &= 0 \end{aligned} \qquad \Rightarrow \qquad \begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(b) Let $u = x - 2$ and $y = v - 4$. Then the differential equations for u, v are:

$$\begin{aligned} u' &= x' = 3(x-2) - (y-4) - 2 + 6 - 4 = 3u - v \\ v' &= y' = 4(x-2) - 2(y-4) + 8 - 8 = 4u - 2v \end{aligned}$$

(c) The eigenvalues and eigenvectors are:

$$\lambda_1 = 2 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \lambda_2 = 2 \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Therefore the solution is:

$$\mathbf{x} = C_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(d) The first equation is: $v = -u' + 3u$. Substitute into the second to get:

$$-u'' + 3u' = 4u - 2(-u' + 3u) = 4u + 2u' - 6u = -2u + 2u' \quad \Rightarrow quadu'' - u' - 2u = 0$$

The characteristic equation is $r^2 - r - 2 = 0$, so $r = -1, 2$ (same as the eigenvalues). Therefore, $u = C_1 e^{2t} + C_2 e^{-t}$, and $v = -u' + 3u = C_1 e^{2t} + 4C_2 e^{-t}$.

(e) Therefore,

$$x(t) = u + 2 = C_1 e^{2t} + C_2 e^{-t}, \qquad y(t) = v + 4 = C_1 e^{2t} + 4C_2 e^{-t} + 4$$

27. Be sure to know the Existence and Uniqueness Theorem for $y' = f(t, y)$ and $y'' + p(t)y' + q(t)y = f(t)$.

    For example, if $y' = y^{1/3}$, $y(0) = 0$, find two solutions to the IVP. Why does this not violate the existence and uniqueness theorem?

    We would typically progress as follows: The equation is separable, so if $y(t) \neq 0$, then:

$$y^{-1/3}\, dy = dt$$

    But $y(0) = 0$. In this case, we see that $y(t) = 0$ is a solution. However, let's go ahead with the previous technique:

$$\frac{3}{2} y^{2/3} = t + C$$

    In this case, we can actually find C so that the initial condition is satisfied, $0 = 0 + C$, and the solution is:

$$y^{2/3} = \frac{2t}{3} \qquad \Rightarrow \qquad y(t) = \left(\frac{2t}{3}\right)^{3/2} \text{ valid for } t \geq 0$$

    The two solutions we found are $y(t) = 0$ and $y(t) = (2t/3)^{3/2}$. This does not violate the Existence and Uniqueness Theorem, since, if $f(t, y) = y^{1/3}$, then

$$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$$

    which does not exist at $y = 0$.

28. (Exercises 3, 7, 9, Section 9.1: See back of book).