Chapter 3, Sect 5

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The Set Up

Find solutions to $L(y) = g(t)$, where

$$L(y) = ay'' + by' + cy$$

The general form of the solution is written as:

$$y(t) = y_h(t) + y_p(t)$$

where y_h solves $L(y) = 0$ (the homogeneous part of the solution), and y_p solves $L(y) = g(t)$ (the particular part of the solution).

Example

Idea: The linear operator $L(y) = ay'' + by' + cy$

Applied to:	Yields:
Polynomials	Polynomials
sin, cos	Combin. of sine and cos.
Exponentials	Exponentials
Products of the above	Products of the above

Example: $L(y) = y'' - y' - 2y$. Then:

$$L(\mathrm{e}^t \sin(3t)) = 3\mathrm{e}^t \cos(3t) - 11\mathrm{e}^t \sin(3t)$$

$$L(t^2) = 2 - 2t - 2t^2$$

and so on.

Second Idea: Superposition

If $L(y) = g_1(t) + g_2(t) + g_3(t)$ (and so on), we can break the solution into that many pieces:

- Let y_{p_1} solve $L(y) = g_1(t)$
- Let y_{p_2} solve $L(y) = g_2(t)$
- Let y_{p_3} solve $L(y) = g_3(t)$

Then $y_p(t) = y_{p_1}(t) + y_{p_2}(t) + y_{p_3}(t)$

Example

Solve: $y'' + 2y' + y = t^2 + e^{2t} - \cos(t)$

- Roots to the char eqn: $r = -1, -1$. Therefore,

$$y_h(t) = e^{-t}(C_1 + C_2 t)$$

Example

- Solve the first particular solution with ansatz: $y_{p_1} = At^2 + Bt + C$. Substituting into the ODE, we get:

$$At^2 + (B + 4A)t + (2A + 2B + C) = t^2$$

  Therefore, $A = 1, B = -4$ and $C = 6$, so that $y_{p_1} = t^2 - 4t + 6$.
- Solve the second: $y_{p_2}(t) = Ae^{2t}$:

$$9Ae^{2t} = e^{2t} \quad \Rightarrow \quad y_{p_2}(t) = \frac{1}{9}e^{2t}$$

- Solve the third: $y_{p_3}(t) = A\cos(t) + B\sin(t)$:

$$-2A\sin(t) + 2B\cos(t) = -\cos(t) \quad \Rightarrow \quad B = -\frac{1}{2}, A = 0$$

  Therefore, $y_{p_3}(t) = -\frac{1}{2}\sin(t)$

Example

In conclusion, given $y'' + 2y' + y = t^2 + e^{2t} - \cos(t)$, the general solution is:

$$y(t) = e^{-t}(C_1 + C_2 t) + t^2 - 4t + 6 + \frac{1}{9}e^{2t} - \frac{1}{2}\sin(t)$$

The Method of Undetermined Coefficients

To find the particular solution, we will guess that its form is the same as $g(t)$ (Also see table in text):

$g_i(t)$ is :	The ansatz for y_{p_i} :
$P_n(t)$	$t^s(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)e^{\alpha t}$	$t^s e^{\alpha t}(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)e^{\alpha t} \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$	$t^s e^{\alpha t}\cos(\beta t)(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0) +$ $t^s e^{\alpha t}\sin(\beta t)(b_n t^n + \ldots + b_2 t^2 + b_1 t + b_0)$

where $s = 0, 1$, or 2.

$$\text{Solve } y'' + 2y' + y = e^{-t}$$

Problem: With the ansatz $y_p = Ae^{-t}$, we get

$$0 = e^{-t}$$

The solution: Multiply the ansatz by t until it is no longer part of the homogeneous solution (e.g., until $L(y_p) \neq 0$).
Since $y_h(t) = C_1e^{-t} + C_2te^{-t}$, we will need to multiply by t^2. Our ansatz is now

$$y_p = At^2e^{-t}$$

Note: Not a full second degree polynomial. Substitution yields $A = 1/2$, so the solution is:

$$y(t) = e^{-t}\left(C_1 + C_2t + \frac{1}{2}t^2\right)$$

Example

Let $y'' - y' - 2y = -4te^t + e^{2t}$. Give your (final) ansatz:

$$r = -1, 2 \quad \Rightarrow \quad y_h(t) = C_1e^{-t} + C_2e^{2t}$$

$$y_{p_1}(t) = (At + B)e^t \quad \Rightarrow \quad (A - 2B)e^t - 2Ate^t = -4te^t$$

so that

$$y_{p_1}(t) = (1 + 2t)e^t$$

$$y_{p_2}(t) = Ate^{2t}$$

Substituting, we find: $3Ae^{2t} = e^{2t}$, so $A = 1/3$. The full solution is

$$y = C_1e^{-t} + C_2e^{2t} + (1 + 2t)e^t + \frac{1}{3}te^{2t}$$

Example

Give the ansatz for the particular part of the solution, if

$$y'' + 2y' + y = te^t \sin(2t)$$

Is this correct?

$$y_p(t) = e^t(At + B)(C\sin(t) + D\cos(t))$$

No! The table says that we need a polynomial for each sine and cosine. That is,

$$y_p(t) = e^t\left((At + B)\sin(2t) + (Ct + D)\cos(2t)\right)$$

And in fact, the full solution to the DE is:

$$e^{-t}(C_1 + C_2 t) + e^t\left(-\frac{1}{8}t + \frac{1}{16}\right)\cos(2t) - \frac{1}{16}e^t\sin(2t)$$

Example

An example we can do by hand: $y'' + y = t - e^t$

$$y_h(t) = C_1\cos(t) + C_2\sin(t)$$

Now the particular solution (break it up):

$$y_{p_1}(t) = At + B \quad \Rightarrow \quad At + B = 1 \quad \Rightarrow \quad y_{p_1}(t) = t$$

And the other part

$$y_{p_2}(t) = Ae^t \quad \Rightarrow \quad 2Ae^t = -e^t$$

Therefore, the full solution is:

$$y(t) = C_1\cos(t) + C_2\sin(t) + t - \frac{1}{2}e^t$$

1. For each DE, give the (final) form of the ansatz. For your convenience, the roots to the characteristic equation are also provided:
   - $y'' + y' = 3t$ with $r = 0, -1$ SOLN: $y_p = t(At + B)$
   - $y'' - 5y' + 6y = t\sin(3t) + \mathrm{e}^{2t}$ with $r = 2, 3$
     $y_p = (At + B)\sin(3t) + (Ct + D)\cos(3t) + At\mathrm{e}^{2t}$
   - $y'' + 2y' + 5y = 3\cos(2t)$ with $r = -1 \pm 2i$ $y_p = A\cos(2t) + B\sin(2t)$
   - $y'' + \omega^2 y = \cos(\omega t)$ with $r = \pm\omega t$ $y_p = t(A\sin(\omega t) + B\cos(\omega t))$
2. Come up with a DE and a forcing function g so that you must multiply your ansatz by t^2.
3. Could you use complex roots for the previous question?