Chapter 3, Sect 6

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Background

Solve: $ay'' + by' + cy = g(t)$.

Recall that the Method of Undetermined Coefficients exploited the linear operator acting on certain classes of functions. If the ansatz was part of the homogeneous solution, multiply by t until it is not.

Today: Variation of Parameters

This method works on the more general form:

$$y'' + p(t)y' + q(t)y = g(t)$$

The method assumes that

$$y_p = u_1(t)y_1(t) + u_2(t)y_2(t)$$

where y_1, y_2 form a fundamental set to the **homogeneous** problem.
Note: We do not know how to get y_1, y_2 except for special cases...

Following through, differentiate in preparation for substitution:

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

We will see this later, but we may assume that $u_1'y_1 + u_2'y_2 = 0$.
Differentiate again:

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2 + u_2y_2''$$

Substitute into the DE:

$$\begin{array}{rlllll} y_p'' & = u_1'y_1' + & u_1y_1'' + & u_2'y_2 + & u_2y_2'' \\ +p(t)y_p' & = & pu_1y_1' + & & pu_2y_2' \\ +q(t)y_p & = & qu_1y_1 + & & qu_2y_2 \\ \hline g(t) & = u_1'y_1' + & 0 + & u_2'y_2' + & 0 \end{array}$$

Summary: Using Variation of Parameters, we assume

$$y_p = u_1 y_1 + u_2 y_2$$

Where y_1, y_2 solve the homogeneous DE. Then u_1, u_2 satisfy the system:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned}$$

NOTE: For exams/quizzes I will give you the system of equations

We solve the system using Cramer's Rule:

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(t) \end{aligned} \quad \Rightarrow u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{W(y_1, y_2)} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{W(y_1, y_2)}$$

From that, we see

$$u_1 = \int \frac{-y_2 g(t)}{W(y_1, y_2)}\, dt \qquad u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt$$

NOTE: Do you need to "add C" to your integrals?

$$u_1 = \int \frac{-y_2 g(t)}{W(y_1, y_2)}\, dt \qquad u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt$$

Use Variation of Parameters to find the general solution:

$$4y'' - 4y' - 8y = 8e^{-t}$$

$$r = -1, 2 \quad \Rightarrow \quad y_1 = e^{-t} \quad y_2 = e^{2t} \quad g(t) = 2e^{-t}$$

We need $W(y_1, y_2)$:

$$W = \begin{vmatrix} e^{-t} & e^{2t} \\ -e^{-t} & 2e^{2t} \end{vmatrix} = 3e^{t}$$

Substituting into the formulas,

$$u_1' = \frac{-e^{2t} 2e^{-t}}{3e^{t}} = -\frac{2}{3} \qquad u_2' = \frac{e^{-t} 2e^{-t}}{3e^{t}} = \frac{2}{3}e^{-3t}$$

Therefore, $u_1(t) = -\frac{2}{3}t$ and $u_2 = -\frac{2}{9}e^{-3t}$.

The particular solution to $4y'' - 4y' - 8y = 8e^{-t}$ is

$$y_p = -\frac{2}{3}te^{-t} - \frac{2}{9}e^{-3t}e^{2t} \Rightarrow y_p = -\frac{2}{3}te^{-t}$$

so the full solution is

$$y(t) = C_1 e^{-t} + C_2 e^{2y} - \frac{2}{3}te^{-t}$$

Note: Using Method of Undetermined Coefficients, we would have guessed the same?

$$u_1 = \int \frac{-y_2 g(t)}{W(y_1, y_2)}\, dt \qquad\qquad u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt$$

Use Variation of Parameters to find the general solution:

$$y'' - 2y' + y = \frac{\mathrm{e}^t}{1 + t^2}$$

SOLUTION: $y_1 = \mathrm{e}^t$, $y_2 = t\mathrm{e}^t$, $W = \mathrm{e}^{2t}$.

$$u_1' = \frac{-t\mathrm{e}^t \frac{\mathrm{e}^t}{(1+t^2)}}{\mathrm{e}^{2t}} = \frac{-t}{1+t^2} \Rightarrow u_1 = -\frac{1}{2}\ln(1+t^2)$$

$$u_2' = \frac{\mathrm{e}^t \frac{\mathrm{e}^t}{(1+t^2)}}{\mathrm{e}^{2t}} = \frac{1}{1+t^2} \Rightarrow u_2 = \tan^{-1}(t)$$

Therefore, the full solution is:

$$y(t) = \mathrm{e}^t\,(C_1 + C_2 t) - \frac{1}{2}\mathrm{e}^t \ln(1+t^2) + t\mathrm{e}^t \tan^{-1}(t)$$

$$u_1 = \int \frac{-y_2 g(t)}{W(y_1, y_2)}\, dt \qquad\qquad u_2 = \int \frac{y_1 g(t)}{W(y_1, y_2)}\, dt$$

Use the Variation of Parameters to solve

$$t^2 y'' - t(t+2)y' + (t+2)y = 2t^3 \qquad y_1(t) = t \quad y_2(t) = t\mathrm{e}^t$$

SOLUTION: $g(t) = 2t$ and $W(y_1, y_2) = t^2\mathrm{e}^t$, so

$$u_1' = \frac{-2t^2\mathrm{e}^t}{t^2\mathrm{e}^t} = -2 \quad \Rightarrow \quad u_1 = -2t$$

Similarly,

$$u_2' = \frac{2t^2}{t^2\mathrm{e}^t} = 2\mathrm{e}^{-t} \quad \Rightarrow \quad u_2 = -2\mathrm{e}^{-t}$$

so

$$y_p = (-2t)(t) + (-2\mathrm{e}^{-t})(t\mathrm{e}^t) = -2t^2 - 2t \quad \Rightarrow y_p(t) = -2t^2$$