Solutions to the Homework
Replaces Section 3.7, 3.8

1. Our text (p. 198) states that

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \approx 1 - \frac{1}{2} \cdot \frac{\gamma^2}{4km}$$

   How was this approximation made? (Hint: Linearize $\sqrt{1-x}$)

   SOLUTION: We linearize $\sqrt{1-x}$ at $x = 0$, which means we use the tangent line to approximate the function. The slope is $\frac{1}{2}(x-1)^{-1/2}(-1)$ which, evaluated at $x = 0$ gives $-1/2$.

   Now in general, $f(x) \approx f(a) + f'(a)(x-a)$ for x close to a, or in this case:

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x$$

   for x small.

2. Show that the period of motion of an undamped vibration of a mass hanging from a vertical spring is $2\pi\sqrt{L/g}$

   SOLUTION: With no damping, $mu'' + ku = 0$ has solution

$$u(t) = A\cos\left(\sqrt{\frac{k}{m}}t\right) + B\sin(\left(\sqrt{\frac{k}{m}}t\right)$$

   so the period is given below. We also note that $mg - kL = 0$, and this equation yields the desired substitution:

$$P = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}} \qquad \text{and} \qquad mg = kL \quad \Rightarrow \quad \frac{k}{m} = \frac{g}{L}$$

3. Using $A = R\cos(\delta), B = R\sin(\delta)$ (and $R = \sqrt{A^2 + B^2}$), and the trig identity:

$$\cos(x-y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

   our text states that we can convert:

$$u = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

   to $u = R\cos(\omega_0 t - \delta)$. Show how this is done with the following:

   (a) $u = 2\cos(3t) + \sin(3t)$

   SOLUTION in detail: We assume that R, δ satisfy the relationships:

$$2 = R\cos(\delta) \qquad 1 = R\sin(\delta)$$

   so that $R^2 = 2^2 + 1^2 = 5$. The angle δ is computed as the argument of the point $(2, 1)$, which is

$$\tan^{-1}(1/2) \approx 0.436$$

   Therefore,

$$2\cos(3t) + \sin(3t) = \sqrt{5}\cos(3t - 0.436)$$

   Fun side note: Try plotting the following in Maple

```
2*cos(3*t)+sin(3*t)-sqrt(5)*cos(3*t-arctan(1/2))
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(b) $u = -2\pi\cos(\pi t) - \pi\sin(\pi t)$
Same idea in this case-

$$R = \pi\sqrt{5} \qquad \text{and } \arctan(1/2) \approx 0.436$$

HOWEVER, be sure to add (or subtract) π since we are in Quadrant III instead of Quadrant I:

$$-2\pi\cos(\pi t) - \pi\sin(\pi t) \approx \pi\sqrt{5}\cos(\pi t - 3.605)$$

(c) $u = 5\sin(t/2) - \cos(t/2)$ The $\tan^{-1}(-1/5) \approx -0.197$ (and add π):

$$\sqrt{26}\cos\left(\frac{t}{2} - 2.944\right)$$

4. Show that in the case of beating,

$$y'' + \omega_0^2 y = a\,\cos(\omega t)$$

the general solution is indeed of the form

$$y(t) = c_1\sin(\omega_0 t) + c_2\cos(\omega_0 t) + \frac{a}{\omega_0^2 - \omega^2}\cos(\omega t)$$

and then, using the formula **TYPO: These should be $(b+a)/2$ and $(b-a)/2$**

$$\cos(a) - \cos(b) = 2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

show that zero initial conditions leads to the solution:

$$y(t) = \frac{2a}{\omega_0^2 - \omega^2}\sin\left(\frac{(\omega_0 - \omega)t}{2}\right)\sin\left(\frac{(\omega_0 + \omega)t}{2}\right)$$

SOLUTION: The roots to the characteristic equation are $r = \pm\omega_0 i$, so by the Method of Undetermined coefficients, if $\omega \neq \omega_0$, the particular part of the solution is:

$$\begin{array}{rl} y_p'' & = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t) \\ +\omega_0^2 y_p & = \omega_0^2(A\cos(\omega t) + B\sin(\omega t)) \\ \hline a\cos(\omega t) & = A(\omega_0^2 - \omega^2)\cos(\omega t) + B(\omega_0^2 - \omega^2)\sin(\omega t) \end{array}$$

Therefore, $B = 0$ and $A = \frac{a}{\omega_0^2 - \omega^2}$, and the full solution to the given ODE is:

$$y(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{a}{\omega_0^2 - \omega^2}\cos(\omega t)$$

Now, with zero ICs ($y(0) = 0$ and $y'(0) = 0$), we have:

$$\begin{array}{rl} 0 & = C_1 + \frac{a}{\omega_0^2 - \omega^2} \\ 0 & = C_2\omega_0 \end{array} \quad \Rightarrow \quad C_2 = 0, C_1 = -\frac{a}{\omega_0^2 - \omega^2}$$

Therefore, the solution is:

$$y(t) = \frac{a}{\omega_0^2 - \omega^2} \left(\cos(\omega t) - \cos(\omega_0 t)\right)$$

And using the given trig substitution (**corrected for the typo**):

$$y(t) = \frac{2a}{\omega_0^2 - \omega^2} \sin\left(\frac{\omega_0 - \omega}{2} t\right) \sin\left(\frac{\omega_0 + \omega}{2} t\right)$$

The reason this form is important is that, if $\omega \approx \omega_0$, the period of the first sine function gets very large, and the amplitude gets very large.

5. (Exercise 7, Section 3.7- Set up by hand, let Maple solve it.)

   NOTE: For an exam or quiz, I would remind you that $g = 32 ft/sec^2$.

   The set up: The spring constant is found by recalling that $mg - kL = 0$. Then, solving for k, we get: $k = mg/L$. In this problem, the weight of 3 pounds is mg, and the length should be in feet: $L = 1/4$, giving $k = 3/(1/4) = 12$. Using $g = 32$ ft/sec^2, the mass is $32m = 3$, of $m = 3/32$. Since there is no damping, we have:

$$\frac{3}{32} u'' + 12u = 0$$

   with initial conditions $u(0) = -\frac{1}{12}$ and $u'(0) = 2$. Going to Maple,

```
Eqn1:=(3/32)*diff(u(t),t$2)+12*u(t)=0;
U:=dsolve({Eqn1, u(0)=-1/12, D(u)(0)=2},u(t));
U1:=rhs(U);
```

   We see that

$$u(t) = -\frac{1}{12} \cos(8\sqrt{2}\, t) + \frac{\sqrt{2}}{8} \sin(8\sqrt{2}\, t)$$

   Now the amplitude and phase are the R, δ that we have solved for previously. In this case,

$$R = \sqrt{\frac{11}{288}} \qquad \delta = \tan^{-1}\left(\frac{3}{\sqrt{2}}\right) + \pi \qquad P = \frac{2\pi}{8\sqrt{2}}$$

6. (Exercise 9, Section 3.7- Set up by hand, let Maple solve it.)

   Exercise 9 is similar- Careful with units (I'll keep units consistent in my questions to you). The damping constant is given using centimeters, so that changes g to 980 cm/sec^2:

$$m = 20 \qquad \gamma = 400 \qquad mg = kL \quad \Rightarrow \quad 20 \cdot 980 = k \cdot 5$$

   Therefore,

$$20u'' + 400u' + 3920u = 0 \qquad u(0) = 2 \qquad u'(0) = 0$$

   Solving in Maple, we get:

```
Eqn1:=20*diff(u(t),t$2)+400*diff(u(t),t)+3920*u(t)=0;
U:=dsolve({Eqn1,u(0)=2,D(u)(0)=0},u(t));
plot(rhs(U),t=0..1);
```

(The answer is the one in the text). The quasi-period (or pseudo-period) is $\frac{2\pi}{4\sqrt{6}}$. The undamped motion ($\gamma = 0$) has period $2\pi/14$, so the ratio is

$$\frac{14}{4\sqrt{6}} \approx 1.43$$

We could also solve the last problem using Maple. Looking at the graph, we estimate the solution to be between $t = 0.35$ and $t = 0.45$, so

```
fsolve(rhs(U)=-0.05, t=0.35..0.45);
```

And the answer is $0.4045\cdots$

7. Find the general solution of the given differential equation:

$$y'' + 3y' + 2y = \cos(t)$$

First, get the homogeneous part of the solution by solving the characteristic equation:

$$r^2 + 3r + 2 = 0 \quad \Rightarrow \quad (r+2)(r+1) = 0 \quad \Rightarrow \quad r = -1, -2$$

Therefore, $y_h(t) = C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-2t}$. You could solve for the particular solution one of two ways- Undetermined Coefficients or using Variation of Parameters. For extra practice, you might try both.

SOLUTION:

$$y_p(t) = A\cos(t) + B\sin(t) \quad y_p' = B\cos(t) - A\sin(t) \quad y_p'' = -A\cos(t) - B\sin(t)$$

so that, looking at the coefficients of $\cos(t)$ and $\sin(t)$, we have:

$$(-A + 3B + 2A)\cos(t) + (-B - 3A + 2B)\sin(t) = \cos(t)$$

Therefore, by Cramer's Rule:

$$\begin{array}{rl} A + 3B & = 1 \\ -3A + B & = 0 \end{array} \quad \Rightarrow \quad A = \frac{1}{10} \quad B = \frac{3}{10}$$

The solution is $C_1 \mathrm{e}^{-t} + C_2 \mathrm{e}^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$

8. What is the *transient solution*? What is the *steady state solution*? (See section 3.8)

On page 208, the transient part of the solution (or transient solution) is the part of the solution that dies off as t gets large- In the example, this was the homogeneous part of the solution (the text called it $u_c(t)$). The steady state solution is the function to which the solution tends as t gets very large (after the transient has died off).

9. Pictured below are the graphs of several solutions to the differential equation:

$$y'' + by' + cy = \cos(\omega t)$$

Match the figure to the choice of parameters:

Choice	b	c	ω
(A)	5	3	1
(B)	1	3	1
(C)	5	1	3
(D)	1	1	3

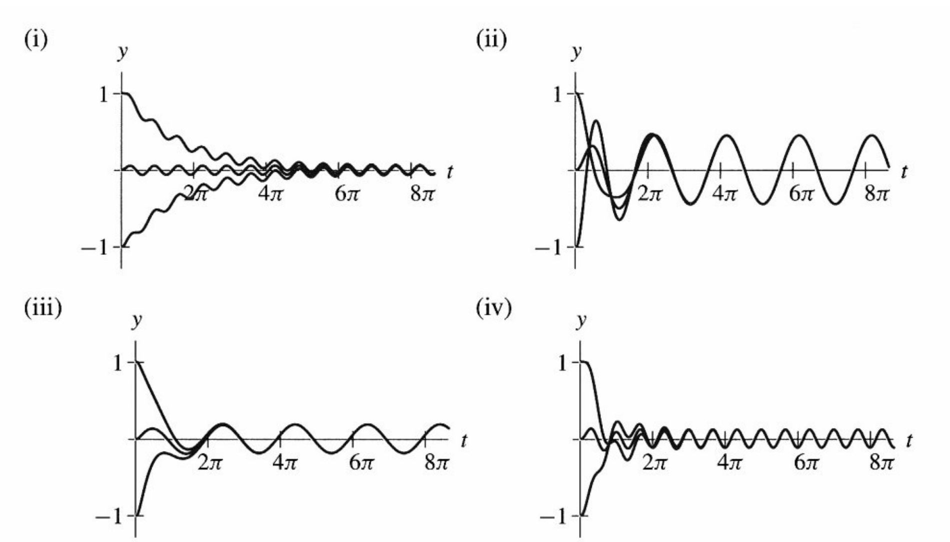

Figure 1: Figures for homework problem 2. Match each figure with the appropriate choice of constants.

We will discuss each situation before locating the curve on the graph.

(a) With $b = 5, c = 3$, we have $r = -0.69$ and -4.3. Therefore, our overall solution will quickly die off to just the particular solution, which will have a period of 2π.

(b) With $b = 1$, $c = 3$, we have $r = -0.5 \pm 1.65i$. Initially, we will have a combination of sinusoidals, but it again will die off (not as quickly as before) to a function whose period is 2π.

(c) With $b = 5, c = 1$, we have something similar to (a), but now the forcing function has period $2\pi/3 \approx 2.09$.

(d) The pseudo-(natural) frequency of the homogeneous part of the equation is $\sqrt{3}2$, but this again dies off leaving a function that is periodic with period about 2.09.

Now, graphs (ii) and (iii) will correspond to (a) and (b) (by the periods). It looks like (iii) will probably correspond to (a), because of how rapidly the solution converges to the particular solution.

So far: (ii) corresponds to (b) and (iii) corresponds to (a).

It would be a safe guess to bet that (i) corresponds to (d) and that leaves (iv) corresponding to (c).

10. Recall that

$$\text{Real}(e^{i\theta}) = \cos(\theta) \qquad \text{Imag}(e^{i\theta}) = \sin(\theta)$$

Show that, given the DE below we can use the ansatz $y_p = Ae^{3ti}$ (the real part),

$$y'' + 4y = 2\cos(3t)$$

and we will get the particular solution,

$$A = -\frac{2}{5} \qquad \Rightarrow \qquad y_p(t) = -\frac{2}{5}\cos(3t)$$

Solution: First differentiate, then substitute into the DE:

$$y_p(t) = Ae^{3it} \quad y_p' = 3iAe^{3it} \quad y_p'' = 9i^2Ae^{3it} = -9Ae^{3it}$$

We notice that $2\cos(3t)$ is the real part of $2e^{3it}$, so:

$$-9Ae^{3it} + 4Ae^{3it} = 2e^{3it} \quad \Rightarrow \quad -5A = 2 \quad \Rightarrow \quad A = -\frac{2}{5}$$

Therefore, taking the real part of $-\frac{2}{5}e^{3it}$ gives us our particular solution.

11. Fill in the question marks with the correct expression:

Given the undamped second order differential equation, $y'' + \omega_0^2 y = A\cos(\omega t)$, we see "beating" if ($|\omega - \omega_0|$ is small) In particular, the longer period wave has a period that gets **longer** as $\omega \to \omega_0$, and its amplitude gets **bigger**

12. Find the solution to $y'' + 9y = 2\cos(3t)$, $y(0) = 0$, $y'(0) = 0$ by first solving the more general equation: $y'' + 9y = 2\cos(at)$, $y(0) = 0$, $y'(0) = 0$, then take the limit of your solution as $a \to 3$.

Solution: First, the homogeneous part of the solution is:

$$y_h(t) = C_1\cos(3t) + C_2\sin(3t)$$

The particular part is (differentiate to substitute into the DE):

$$y_p = A\cos(at) + B\sin(at) \qquad y_p'' = -Aa^2\cos(at) - Ba^2\sin(at)$$

And,

$$y_p'' + 9y_p = A\left(-a^2 + 9\right)\cos(at) + B\left(-a^2 + 9\right) = 2\cos(at) \quad \Rightarrow \quad A = \frac{2}{9 - a^2} \quad B = 0$$

The particular part of the solution is:

$$y_p(t) = \frac{2}{9 - a^2}\cos(at)$$

Put everything together to solve with the initial conditions, $y(0) = 0$ and $y'(0) = 0$:

$$y = C_1 \cos(3t) + C_2 \sin(3t) + \frac{2}{9-a^2}\cos(at) \quad \Rightarrow \quad 0 = C_1 + \frac{2}{9-a^2} \quad \Rightarrow \quad C_1 = -\frac{2}{9-a^2}$$

and, using $y'(0) = 0$,

$$0 = 0 + 3C_2 + 0 \quad \Rightarrow \quad C_2 = 0$$

Therefore, the overall solution to the IVP is:

$$y(t) = \frac{2}{9-a^2}\left(\cos(at) - \cos(3t)\right)$$

Take the limit as $a \to 3$ using L'Hospital's rule:

$$\lim_{a\to 3} \frac{2(\cos(at)-\cos(3t))}{9-a^2} = \lim_{a\to 3}\frac{-2t\sin(at)}{-2a} = \frac{-2t\sin(3t)}{(-2)(3)} = \frac{1}{3}t\sin(3t)$$

13. (Exercise 6, Section 3.8: Do this one by hand.)

Set up the constants first- Use meters, kilograms and seconds so that they all line up- In that case,

$$m = 5 \qquad k(0.1) = 5(9.8) \Rightarrow k = 490 \quad \text{and} \quad 0.04\gamma = 2 \quad \Rightarrow \quad \gamma = 50$$

Therefore,

$$5u'' + 50u' + 490u = 10\sin\left(\frac{t}{2}\right)$$

with $u(0) = 0$ and $u'(0) = 0.03$.

14. (Exercise 10, Section 3.8: Verify your solution with Maple).

We see that $mg = 8$ with $g = 32$, and $mg - kL = 0$ (use feet),

$$8 - \frac{k}{2} = 0 \quad \Rightarrow \quad k = 16$$

And $m = \frac{8}{32} = \frac{1}{4}$ gives:

$$\frac{1}{4}u'' + 16u = 8\sin(8t) \quad \Rightarrow \quad u'' + 64u = 32\sin(8t)$$

Using the method of undetermined coefficients, the particular part of the solution is found by guessing, then substitute the guess into the DE:

$$u_p = t(A\cos(8t) + B\sin(8t)) \quad \Rightarrow$$

$$\begin{array}{rl} u_p'' & = (16B - 64At)\cos(8t) + (-16A - 64Bt)\sin(8t) \\ 64u_p & = 64At\cos(8t) + 64Bt\sin(t) \\ \hline 32\sin(8t) & = 16B\cos(8t) - 16A\sin(8t) \end{array}$$

From which we get $A = -2$ and $B = 0$ so that the overall solution so far is:

$$u(t) = C_1\cos(8t) + C_2\sin(8t) - 2t\cos(8t)$$

Using the initial conditions $u(0) = \frac{1}{4}$ and $u'(0) = 0$, we get:

$$\frac{1}{4} = C_1 = C_2$$

For the remainder, we can use Maple.

15. (Exercise 17, Section 3.8: Set up by hand, have Maple compute the amplitude in terms of ω and plot.)

    Here are the Maple commands:

```
Eqn1:=diff(u(t),t$2)+(1/4)*diff(u(t),t)+2*u(t)=2*cos(w*t);
U1:=dsolve(Eqn1,u(t));
A:=coeff(rhs(U1),cos(w*t)); B:=coeff(rhs(U1),sin(w*t));
R:=simplify(sqrt(A^2+B^2));
plot(R,w=0..4);
maximize(R,w=1..2);
```