Chapter 3 Sample Questions

These are not meant to be comprehensive, they are meant to give you questions out of the context of any particular section of the textbook. Be sure you understand the homework problems and quiz questions.
The following formulas will be given to you:

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g(t) & y_2' \end{vmatrix}}{W(y_1, y_2)} \qquad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g(t) \end{vmatrix}}{W(y_1, y_2)}$$

and the trig formula:

$$A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta)$$

where $R = \sqrt{A^2 + B^2}$, $tan(\delta) = B/A$.

1. State the Existence and Uniqueness theorem for linear, second order differential equations (non-homogeneous is the most general form):

2. If the $W(y_1, y_2) = t^2$, can y_1, y_2 be two solutions to $y'' + p(t)y' + q(t)y = 0$? Explain.

3. Construct the operator associated with the differential equation: $y' = y^2 - 4$. Is the operator linear? Show that your answer is true by using the definition of a linear operator.

4. Find the solution to the initial value problem:

$$u'' + u = \begin{cases} 3t & \text{if } 0 \le t \le \pi \\ 3(2\pi - t) & \text{if } \pi < t < 2\pi \\ 0 & \text{if } t \ge 2\pi \end{cases} \qquad u(0) = 0 \quad u'(0) = 0$$

5. Solve: $u'' + \omega_0^2 u = F_0 \cos(\omega t)$, $\quad u(0) = 0 \quad u'(0) = 0$ if $\omega \neq \omega_0$ using the Method of Undetermined Coefficients.

6. Compute the solution to: $u'' + \omega_0^2 u = F_0 \cos(\omega_0 t) \qquad u(0) = 0 \quad u'(0) = 0$ two ways:

   - Start over, with Method of Undetermined Coefficients
   - Take the limit of your answer from Question 6 as $\omega \to \omega_0$.

7. For the following question, recall that the acceleration due to gravity is 32 ft/sec^2.

   An 8 pound weight is attached to a spring from the ceiling. When the weight comes to rest at equilibrium, the spring has been stretched 2 feet. The damping constant for the system is 1−lb-sec/ft. If the weight is raised 6 inches above equilibrium and given an upward velocity of 1 ft/sec, find the equation of motion for the weight. Write the solution as $R\cos(\omega t - \delta)$, if possible.

8. Given that $y_1 = \frac{1}{t}$ solves the differential equation:

$$t^2 y'' - 2y = 0$$

Find the fundamental set of solutions.

9. Suppose a mass of 0.01 kg is suspended from a spring, and the damping factor is $\gamma = 0.05$. If there is no external forcing, then what would the spring constant have to be in order for the system to *critically damped? underdamped?*

10. Give the full solution, using any method(s). If there is an initial condition, solve the initial value problem.

    (a) $y'' + 4y' + 4y = t^{-2}\mathrm{e}^{-2t}$

    (b) $y'' - 2y' + y = t\mathrm{e}^{t} + 4$, $y(0) = 1$, $y'(0) = 1$.

    (c) $y'' + 4y = 3\sin(2t)$, $y(0) = 2$, $y'(0) = -1$.

    (d) $y'' + 9y = \sum_{m=1}^{N} b_m \cos(m\pi t)$

11. Rewrite the expression in the form $a + ib$: (i) 2^{i-1} (ii) $\mathrm{e}^{(3-2i)t}$ (iii) $\mathrm{e}^{i\pi}$

12. Find a linear second order differential equation with constant coefficients if

$$y_1 = 1 \qquad y_2 = \mathrm{e}^{-t}$$

form a fundamental set, and $y_p(t) = \frac{1}{2}t^2 - t$ is the particular solution.

13. Determine the longest interval for which the IVP is certain to have a unique solution (Do not solve the IVP):

$$t(t-4)y'' + 3ty' + 4y = 2 \qquad y(3) = 0 \quad y'(3) = -1$$

14. Let $L(y) = ay'' + by' + cy$ for some value(s) of a, b, c.

    If $L(3\mathrm{e}^{2t}) = -9\mathrm{e}^{2t}$ and $L(t^2 + 3t) = 5t^2 + 3t - 16$, what is the particular solution to:

$$L(y) = -10t^2 - 6t + 32 + \mathrm{e}^{2t}$$

15. Show that, using the substitution $x = \ln(t)$, then the differential equation:

$$4t^2 y'' + y = 0$$

becomes a differential equation with constant coefficients.

Solve it.

16. Compute the Wronskian of two solutions of the given DE without solving it:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

17. Given $t^2y'' - 2y = 0$ and $y_1 = 1/t$, find y_2 by reduction of order.

18. If $y'' - y' - 6y = 0$, with $y(0) = 1$ and $y'(0) = \alpha$, determine the value(s) of α so that the solution tends to zero as $t \to \infty$.

19. Give the general solution to $y'' + y = \frac{1}{\sin(t)} + t$

20. Given each of the DEs below, mark if it corresponds to beating, resonance, periodic.

    (a) $y'' + 36y = 5\cos(6t)$

    (b) $y'' + 36y = 5\cos(5t)$

    (c) $y'' + 36y = 0$

21. Describe the graph of a typical solution to:

    (a) $y'' + y' + 36y = 5\cos(5t)$

    (b) $y'' + y' + 36y = 0$

    (c) $y'' + 12y' + 36y = 0$

    (In other words, if I gave you three graphs, could you work out which belongs to which?)