Chapter 3, Theory

The goal of the theory was to establish the structure of solutions to the second order DE:

$$y'' + p(t)y' + q(t)y = g(t)$$

We saw that two functions form a fundamental set of solutions to the homogeneous DE if the Wronskian is not zero (at the initial value of time).

1. Vocabulary: Operator, Linear Operator, general solution, fundamental set of solutions.
2. Theorems:
   - The Existence and Uniqueness Theorem for $y'' + p(t)y' + q(t)y = g(t)$.
   - Principle of Superposition.
   - Abel's Theorem.

     If y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$, then the Wronskian is either always zero or never zero on the interval for which the solutions are valid.

     That is because the Wronskian may be computed as:

     $$W(y_1, y_2)(t) = Ce^{-\int p(t)\, dt}$$

   - The Fundamental Set of Solutions: $y'' + p(t)y' + q(t)y = 0$

     We can guarantee that we can always find a fundamental set of solutions. We did that by appealing to the Existence and Uniqueness Theorem for the following two initial value problems:

     - y_1 solves $y'' + p(t)y' + q(t)y = 0$ with $y(t_0) = 1, y'(t_0) = 0$
     - y_2 solves $y'' + p(t)y' + q(t)y = 0$ with $y(t_0) = 0, y'(t_0) = 1$
3. The Structure of Solutions to $y'' + p(t)y' + q(y)y = g(t)$, $y(t_0) = y_0, y'(t_0) = v_0$

   Given a fundamental set of solutions to the homogeneous equation, y_1, y_2, then there is a solution to the initial value problem, written as:

   $$y(t) = C_1y_1(t) + C_2y_2(t) + y_p(t)$$

   where $y_p(t)$ solves the non-homogeneous equation.

   In fact, if we have:

   $$y'' + p(t)y' + q(t)y = g_1(t) + g_2(t) + \ldots + g_n(t)$$

   we can solve by splitting the problem up into smaller problems:

- y_1, y_2 form a fundamental set of solutions to the homogeneous equation.
- y_{p_1} solves $y'' + p(t)y' + q(t)y = g_1(t)$
- y_{p_2} solves $y'' + p(t)y' + q(t)y = g_2(t)$
  and so on..
- y_{p_n} solves $y'' + p(t)y' + q(t)y = g_n(t)$

and the full solution is:

$$y(t) = C_1 y_1 + C_2 y_2 + y_{p_1} + y_{p_2} + \ldots + y_{p_n}$$

Analysis of the Oscillators

Given

$$mu'' + \gamma u' + ku = F(t)$$

we should be able to determine the constants from a given setup for a spring-mass system.

1. Unforced ($F(t) = 0$)
   (a) No damping: Natural frequency is $\sqrt{k/m}$
   (b) With damping: Underdamped, Critically Damped, Overdamped
2. Forced
   (a) With damping: Transient and steady-state solutions.
   (b) With no damping, Periodic forcing: Beating and Resonance

NOTE: Trig identities used here will be provided.