Quiz 2 Solutions

1. Set up and solve the IVP corresponding to the amount of salt in the tank at time t.

   A tank originally contains 100 gallons of water with a concentration of 1 pound of salt per gallon. Brine is pouring into the tank at a rate of 4 gallons per minute, and the incoming brine has a concentration of 1/2 pound of salt per gallon. The well-mixed solution is draining out of the tank at a rate of 3 gallons per minute.

   SOLUTION: Let $Q(t)$ be the amount of salt in the tank at time t. Then

   $$Q' = 2 - \frac{3}{100+t}Q \qquad Q(0) = 100$$

   The DE is a linear differential equation, and the integrating factor is $(100+t)^3$. The solution is:

   $$Q(t) = \frac{1}{2}(100+t) + \frac{100^4}{2(100+t)^3}$$

2. (a) Find the (largest) interval on which the solution is valid (from the Existence and Uniqueness Theorem, if possible). HINT: Do not solve the IVP unless you have to.

   $$(4-t^2)y' + 2ty = 3t^2 \qquad y(-5) = -1$$

   SOLUTION: Put the equation in standard form for a linear DE:

   $$y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$$

   The expression for $p(t)$ has vertical asymptotes at $t = \pm 2$. Looking at the initial time ($t_0 = -5$), we should choose the interval $(-\infty, -2)$ for the interval on which the solution is valid.

   (b) Compute the given integral: $\int \frac{1-x}{1+x^2}\, dx$

   SOLUTION:

   $$\int \frac{1}{1+x^2}\, dx - \int \frac{x}{1+x^2} = \tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C$$

   (We don't need absolute value signs with the log; $1+x^2$ is always positive).

   (c) Compute the given integral: $\int \mathrm{e}^{-2t}\sin(3t)\, dt$

   SOLUTION: Use the table to integrate by parts twice- You should get:

   $$-\frac{\mathrm{e}^{-2t}}{13}\left(3\cos(3t) + 2\sin(3t)\right) + C$$

3. (a) Find all value(s) of r so that $y = t^r$ solves the differential equation:

   $$2t^2y'' + 3ty' - y = 0$$

   SOLUTION: Substitute $y = t^r, y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$ into the DE. Factor out t^r, and you should get:

   $$t^r(2r^2 + r - 1) = 0$$

   which should be true for all t (so we just need to solve the quadratic). The values of r are $1/2$ and -1.

(b) Find all value(s) of k so that e^{kt} solves the differential equation:

$$y'' + y' - 6y = 0$$

SOLUTION: Similarly, substitute $y = e^{kt}$, $y' = ke^{kt}$ and $y'' = k^2 e^{kt}$ into the DE. You should find that

$$k^2 + k - 6 = 0 \quad \Rightarrow \quad k = 3, -2$$

4. Let $y' = y(y-2)$ with $y(0) = 1$. Treating this as a *Bernoulli Equation* (See pg. 77), solve the IVP (explicitly).

SOLUTION: Rewriting the IVP, we get: $y' = y^2 - 2y$ or $y' + 2y = y^2$, which is Bernoulli. Dividing by y^2, we have:

$$\frac{y'}{y^2} + 2 \cdot \frac{1}{y} = 1 \quad \Rightarrow \quad v = \frac{1}{y}, v' = -\frac{y'}{y^2} \quad \Rightarrow \quad -v' + 2v = 1 \quad \Rightarrow \quad v(t) = \frac{1}{2} + Ae^{2t}$$

Now back-substitute:

$$\frac{1}{y} = \frac{1}{2} + Ae^{2t} \quad \Rightarrow \quad y(t) = \frac{2}{1 + e^{2t}}$$

5. Let $y' = y(y-2)$ (as before). Draw the phase plot (or equivalently, the phase diagram), and the corresponding direction field. On the phase plot, indicate where y is increasing/decreasing and concave up/concave down. On the direction field, indicate the equilibrium solutions and whether or not the equilibria are stable or unstable.

SOLUTION: You should find that $y = 0$ is stable, $y = 2$ is unstable, and:

For $y < 0$	y is increasing
	y is CD
$0 < y < 1$	y is decreasing
	y is CU
$1 < y < 2$	y is decreasing
	y is CD
$y > 2$	y is increasing
	y is CU