Quiz 10 Solutions

1. Modify the Maple files online (`LinearSystems02.mw`) so you can plot the direction field for a nonlinear system of equations (rather than a system of linear equations).

   In particular, plot the direction field for:

$$\begin{aligned} x' &= 2xy \\ y' &= -x^2 + 3y^2 \end{aligned}$$

   On the direction field, also plot sample solution curves for $-1 \le t \le 8$ for the following sets of initial conditions:

$$\begin{aligned} x(0) &= 1 \qquad & x(0) &= 1/2 \qquad & x(0) &= -1/2 \\ y(0) &= 1 \qquad & y(0) &= 1/4 \qquad & y(0) &= -2/5 \end{aligned}$$

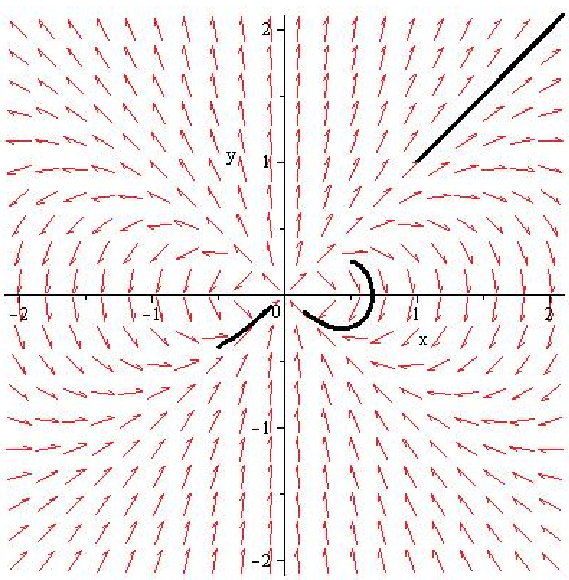
   Print your Maple file and attach to your solutions.

   SOLUTION: Here are the relevant Maple commands:

```
Sys:=diff(x(t),t)=2*x(t)*y(t),
diff(y(t),t)=-x(t)^2+3*y(t)^2;

ICs:=[[x(0)=1,y(0)=1],[x(0)=1/2,y(0)=1/4],[x(0)=-1/2,y(0)=-2/5]];

DEplot({Sys},[x(t),y(t)],t=0..5,x=-2..2,y=-2..2,ICs,linecolor=black);
```



2. (By hand) Solve the system of differential equations by solving the expression for dy/dx (HINT: It is homogeneous, but solve it as a Bernoulli equation with n a negative number!)

SOLUTION:

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} = \frac{3}{2x}y - 2xy^{-1} \quad \Rightarrow \quad y' - \frac{3}{2x}y = -\frac{1}{2}xy^{-1}$$

Divide by y^{-1} (or multiply by y) to get:

$$y'y - \frac{3}{2x}y^2 = -\frac{1}{2}x \quad \Rightarrow \quad 2yy' - \frac{3}{x}y^2 = -x$$

Let $v = y^2$. Then $v' = 2yy'$, and:

$$v' - \frac{3}{x}v = -x$$

This is linear, with integrating factor

$$\mathrm{e}^{-3\int \frac{1}{x}\,dx} = x^{-3} = \frac{1}{x^3}$$

Multiply both sides by the integrating factor and solve:

$$\left(\frac{v}{x^3}\right)' = -x^{-2} \quad \Rightarrow \quad \frac{v}{x^3} = \frac{1}{x} + C \quad \Rightarrow \quad v = x^2 + Cx^3$$

Backsubtitute, to get that: $y^2 = x^2 + Cx^3$

3. Do exercises 1, 4, 7 on p. 494-495, with the following (slightly modified) instructions:

   (a) Find the eigenvalues and eigenvectors (by hand).

   (b) Write the general solution to the system (by hand).

   (c) Classify the origin using the Poincaré Diagram (by hand of course).

   (d) Sketch the direction field and several solution curves using Maple (one graph for each problem).

p. 494, 1

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 - \lambda - 2 = 0$$

so $\lambda = -1, 2$. For $\lambda = -1$, we have:

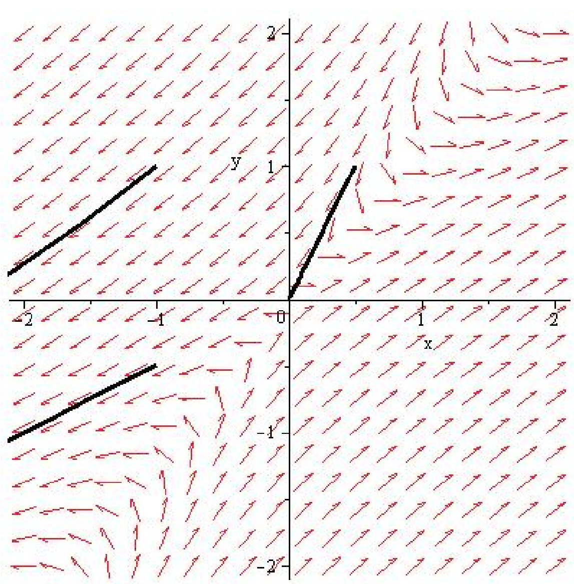
$$\begin{bmatrix} 3+1 & -2 \\ 2 & -2+1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad 2v_1 - v_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

For $\lambda = 2$, we have:

$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 - 2v_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The solution to the differential equation is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \mathrm{e}^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



p. 494, 4

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 + 6\lambda + 9 = 0$$

so $\lambda = -3$ (doubled). The eigenvector $\mathbf{v}$ is found the usual way:

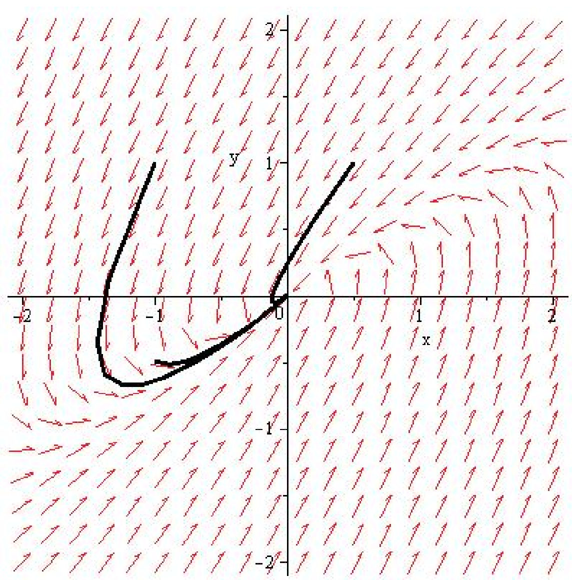
$$\begin{bmatrix} 1+3 & -4 \\ 4 & -7+3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 - v_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For the generalized eigenvector $\mathbf{w}$, solve:

$$\begin{array}{ll} 4w_1 - 4w_2 & = 1 \\ 4w_1 - 4w_2 & = 1 \end{array} \quad \Rightarrow \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 0 \end{bmatrix}$$

The solution to the differential equation is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^{2t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \right)$$



p. 494, 7

$$A = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 - 2\lambda + 5 = 0$$

so $\lambda = 2 \pm i$. For $\lambda = 2 + i$, we have:

$$\begin{bmatrix} (1-i) & -2 \\ 4 & (-3-i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad (1-i)v_1 - 2v_2 = 0 \quad \Rightarrow \quad \begin{bmatrix} 2 \\ 1-i \end{bmatrix}$$

Now compute $e^{\lambda t}\mathbf{v}$:

$$e^{(2+i)t} \begin{bmatrix} 2 \\ 1-i \end{bmatrix} = e^{2t}(\cos(t)+i\sin(t)) \begin{bmatrix} 2 \\ 1-i \end{bmatrix} = e^{2t} \begin{bmatrix} 2\cos(t) + 2i\sin(t) \\ (\cos(t) + \sin(t) + i(\sin(t) - \cos(t)) \end{bmatrix}$$

The solution to the differential equation is:

$$\mathbf{x}(t) = C_1 e^{2t} \begin{bmatrix} 2\cos(t) \\ (\cos(t) + \sin(t) \end{bmatrix} + C_2 e^{2t} \begin{bmatrix} 2\sin(t) \\ \sin(t) - \cos(t) \end{bmatrix}$$