Study Guide: Exam 1, Math 244

The exam covers material from Chapters 1 and 2 (up to 2.6), and will 50 minutes in length. You may not use the text or notes. You may use a scientific calculator (graphing calculators are OK), but NOT a calculator that can perform symbolic manipulation (if you're not sure about your calculator, ask before the day of the exam!). Typically, the use of the calculator on the exam is intended only for numerical approximations and arithmetic.

Fundamental Ideas

A differential equation defines a family of functions (these are the solutions), and as such, ODEs provide a powerful tool for modeling.

When we solve an ODE, we not only want to get an analytic solution, but we also want to understand graphical analysis (direction fields, phase plots) and we want to be able to analyze the solution- Is the solution unique? What is its behavior over the long term?, etc.

Vocabulary

You should what these terms mean: differential equation, ordinary differential equation, partial differential equation, order of a differential equation, linear differential equation, equilibrium solution, isocline, direction field

And be able to identify the following types of DEs: Linear, separable, homogeneous, autonomous. You might also keep Bernoulli equations in mind as an extra tool.

The Existence and Uniqueness Theorem

Know these!

1. Linear: $y' + p(t)y = g(t)$ at (t_0, y_0):

   If p, g are continuous on an interval I that contains t_0, then there exists a unique solution to the initial value problem and that solution is valid for all t in the interval I.

2. General Case: $y' = f(t, y)$, (t_0, y_0):

   (a) If f is continuous on a small rectangle containing (t_0, y_0), then there exists a solution to the initial value problem.

   (b) If $\partial f/\partial y$ is continuous on that small rectangle containing (t_0, y_0), then that solution is unique.

   (c) We can only guarantee that the solution persists on a small interval about (t_0, y_0). To find the full interval, we need to actually solve the initial value problem.

Graphical Analysis

1. Be able to use a direction field to analyze the behavior of solutions to general first order equations. Be able to construct simple direction fields using isoclines.

2. Special Case: **Autonomous DEs:** The main idea here is to be able to graph the phase plot, $y' = f(y)$ in the (y, y') plane and be able to translate the information from this graph to the direction field, the (t, y) plane.

   Here is a summary of that information:

In Phase Diagram:	In Direction Field:
y intercepts	Equilibrium Solutions
$+$ to $-$ crossing	Stable Equilibrium
$-$ to $+$ crossing	Unstable Equilibrium
$y' > 0$	y increasing
$y' < 0$	y decreasing
y' and df/dy same sign	y is concave up
y' and df/dy mixed	y is concave down

Recall that we also looked at a theorem about determining the stability of an equilibrium solution using the sign of df/dy, and determining a formula for y'' given $y' = f(y)$.

Analytic Solutions

- Linear: $y' + p(t)y = g(t)$. Use the integrating factor: $e^{\int p(t)\,dt}$
- Separable: $y' = f(y)g(t)$. Separate variables: $(1/f(y))\,dy = g(t)\,dt$
- Solve by substitution:
  - Homogeneous: $\frac{dy}{dx} = F(y/x)$. Substitute $v = y/x$ (and get the expression for dv/dx as well).
  - Bernoulli: $y' + p(t)y = y^n$ Divide by y^n, let $w = y^{1-n}$ and it becomes linear.

  NOTE: I'll give a hint for these.

- Exact: $M(x, y) + N(x, y)\frac{dy}{dx}$, where $N_x = M_y$.

  Solution: Set $f_x(x, y) = M(x, y)$. Integrate w/r to x. Check that $f_y = N(x, y)$, and add a function of y if necessary.

  NOTE: I'll give an integrating factor, if necessary.

Models

Be familiar with (be able to construct) the following models:

Exponential growth, Logistic growth, Free fall, Newton's Law of Cooling, Tank Mixing.

For any physics problems, values of constants (like g) would be given to you.