# Solutions to the Review Questions

## Short Answer/True or False

1. True or False, and explain: If $y' = y + 2t$, then $0 = y + 2t$ is an equilibrium solution.

   False: (a) Equilibrium solutions are only defined for *autonomous* differential equations, (b) This is an isocline for a slope of zero, and (c) $y = -2t$ is not a solution.

2. 
   - The Existence and Uniqueness Theorem for linear first order initial value problems (IVPs).
     Let $y' + p(t)y = g(t)$ with $y(t_0) = y_0$.
     If $p, g$ are continuous on an open interval $I$ containing $t_0$, then a unique solution exists to the IVP. In addition, the solution is valid on $I$.
     (Note: The interval $I$ is a single (connected) interval, not two or more intervals).
   - The general Existence and Uniqueness Theorem for first order initial value problems (IVPs).
     Let $y' = f(t, y)$ with $y(t_0) = y_0$.
     If $f$ is continuous on an open rectangle containing $(t_0, y_0)$, then a solution exists.
     If $\partial f/\partial y$ is continuous on an open rectangle containing $(t_0, y_0)$, then the solution is unique.
     There is a small interval about $t_0$ on which the (unique) solution will exist, but we cannot predict what it will be in advance- We need to actually solve the IVP.

3. Let $y' = f(y)$. It is possible to have two stable equilibrium with no other equilibrium between them.

   It is, but only if $f$ is not continuous. If $f$ is continuous (which is a normal assumption on $f$), then it is not possible (draw a picture in the phase plane and you'll see why- FYI, it is a consequence of the Intermediate Value Theorem).

4. 
   - What is a *linear* first order differential equation.
     A linear first order DE is any DE that can be expressed as:
     $$y' + p(t)y = g(t)$$
   - What is an $n^{\text{th}}$ order differential equation?
     The order of a differential equation refers to the integer of the highest derivative. An $n^{\text{th}}$ order DE would have an $n^{\text{th}}$ derivative as the highest derivative.

5. What's the difference between:

   "The domain of $y(t)$" and "The time interval for which $y(t)$ is a solution to the DE"?

   When we talk about the time interval on which a solution is valid, the interval must be a single (connected) interval. A domain can be any combination of points, intervals, etc.

6. Let $\frac{dy}{dt} = 1 + y^2$. Then the solution will be valid for all $t$.

   False. Solving the DE:
   $$\int \frac{1}{1+y^2}\,dy = \int dt \quad \Rightarrow \quad \tan^{-1}(y) = t + C \quad \Rightarrow \quad y = \tan(t + c)$$

   The tangent function has vertical asymptotes at $t + c = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$, so there will be a strip of time of length $\pi$ on which the solution will be valid (but no more).

7. To solve $y' = y^{1/3}$, we separate variables:
   $$y^{-1/3}\,dy = dt$$

   Before going further, it is good practice to note that the previous step is valid, *as long as* $y \neq 0$. The case that $y = 0$ can be taken separately- In fact, we see that $y(t) = 0$ is an equilibrium solution that satisfies the initial condition.

Going on, we integrate:

$$\frac{3}{2}y^{2/3} = t + C_1 \quad \Rightarrow \quad y^{2/3} = \frac{2}{3}t + C_2 \quad \Rightarrow \quad y = \left(\frac{2}{3}t + C_2\right)^{3/2}$$

We can solve for $C_2$ using the initial condition: $0 = C_2$, so that

$$y = \left(\frac{2t}{3}\right)^{3/2}$$

We can verify that this is indeed a solution by substituting it back into the DE (not necessary; just a way of double-checking yourself):

$$y' = \frac{3}{2}\left(\frac{2t}{3}\right)^{1/2} \cdot \frac{2}{3} = \left(\frac{2t}{3}\right)^{1/2}$$

And on the other hand,

$$y^{1/3} = \left[\left(\frac{2t}{3}\right)^{3/2}\right]^{1/3} = \left(\frac{2t}{3}\right)^{1/2}$$

Therefore, this is indeed a second solution to the IVP.

Of course, the Existence and Uniqueness Theorem cannot be "violated" since it is a theorem, but in this case, the *conditions* are not met:

$$y' = f(t, y) \quad \Rightarrow \quad f(t, y) = y^{1/3}$$

In this case, $f$ is continuous at $(0, 0)$ but $\partial f/\partial y$ is not.

## Solve:

1. $\dfrac{dy}{dx} = \dfrac{x^2 - 2y}{x}$

   Linear: $y' + \frac{2}{x}y = x$ Solve with an integrating factor of $x^2$ to get:

   $$y = \frac{1}{4}x^2 + \frac{C}{x^2}$$

2. $(x + y)\,dx - (x - y)\,dy = 0$. Hint: Let $v = y/x$.

   Given the hint, rewrite the DE:

   $$\frac{dy}{dx} = \frac{x + y}{x - y} = \frac{1 + (y/x)}{1 - (y/x)} = \frac{1 + v}{1 - v}$$

   With the substitution $xv = y$, we get the substitution for $dy/dx$:

   $$v + xv' = y'$$

   So that the DE becomes:

   $$v + xv' = \frac{1 + v}{1 - v} \quad \Rightarrow \quad xv' = \frac{1 + v}{1 - v} - v = \frac{1 + v - v(1 - v)}{1 + v} = \frac{1 + v^2}{1 - v}$$

   The equation is now separable:

   $$\frac{1 - v}{1 + v^2}\,dv = \frac{1}{x}\,dx \quad \Rightarrow \quad \int \frac{1}{1 + v^2}\,dv - \int \frac{v}{1 + v^2}\,dv = \ln|x| + C$$

   Therefore,

   $$\tan^{-1}(v) - \frac{1}{2}\ln(1 + v^2) = \ln|x| + C$$

   Lastly, back-substitute $v = y/x$.

3. $\dfrac{dy}{dx} = \dfrac{2x+y}{3+3y^2-x} \qquad y(0)=0.$

   This is exact. The solution is, with $y(0)=0$,

$$-x^2 - xy + 3y + y^3 = 0$$

4. $\dfrac{dy}{dx} = -\dfrac{2xy+y^2+1}{x^2+2xy}$

   This is exact. The solution is: $x^2y + xy^2 + x = c$

5. $\frac{dy}{dt} = 2\cos(3t) \qquad y(0)=2$

   This is linear and separable. $y(t) = \frac{2}{3}\sin(3t) + 2$, and the solution is valid for all time.

6. $y' - \frac{1}{2}y = 0 \qquad y(0) = 200$. State the interval on which the solution is valid.

   This is linear and separable. As a linear equation, the solution will be valid on all $t$ (since $p(t) = -\frac{1}{2}$). The solution is $y(t) = 200\mathrm{e}^{(1/2)t}$

7. This is separable (or Bernoulli):

$$\int y^{-2}\,dy = \int (1-2x)\,dx \quad \Rightarrow \quad -\frac{1}{y} = x - x^2 + C$$

   Put in the initial condition (IC): $6 = 0 + C$. Now finish solving explicitly:

$$y(x) = \frac{1}{x^2 - x - 6} = \frac{1}{(x-3)(x+2)}$$

   The solution is valid on the interval $(-2, 3)$.

8. $y' - \frac{1}{2}y = \mathrm{e}^{2t} \qquad y(0) = 1$

   This is linear (but not separable). $y(t) = \frac{2}{3}\mathrm{e}^{2t} + \frac{1}{3}\mathrm{e}^{(1/2)t}$

9. $y' = \frac{1}{2}y(3-y)$

   Autonomous (and separable). Integrate using partial fractions:

$$\int \frac{1}{y(3-y)}\,dy = \frac{1}{2}\int\,dt$$

   Simplify your answer for $y$ by dividing numerator and denominator appropriately to get:

$$y(t) = \frac{3}{(1/A)\mathrm{e}^{-(3/2)t} + 1}$$

10. $\sin(2t)\;dt + \cos(3y)\;dy = 0$

    Separable (and/or exact): $\frac{-1}{2}\cos(2t) + \frac{1}{3}\sin(3y) = C$

11. $y' = xy^2$

    Separable: $y = \dfrac{1}{-(1/2)x^2 - C}$

12. $\dfrac{dy}{dt} = \mathrm{e}^{t+y}$

    Separable: $y' = \mathrm{e}^t\mathrm{e}^y$, so:

$$\int \mathrm{e}^{-y}\,dy = \int \mathrm{e}^t\,dt$$

    and $-\mathrm{e}^{-y} = \mathrm{e}^t + C$

13. $\dfrac{dy}{dx} + y = \dfrac{1}{1+e^x}$

Linear: $y' + y = 1/(1+e^x)$, and the I.F. is $e^x$. Therefore,

$$(e^x y) = \int \frac{e^x}{1+e^x}\, dx$$

To integrate, use $u, du$ substitution. The solution is then:

$$y = \frac{\ln(1+e^x) + C}{e^x}$$

14. $(t^2y + ty - y)\, dt + (t^2y - 2t^2)\, dy = 0$

Does not seem to be exact. Try separating variables:

$$\frac{dy}{dt} = \frac{-y(t^2+t+1)}{(y-2)\cdot t^2}$$

so:

$$\frac{y-2}{y}\, dy = -\left(1 + \frac{1}{t} + t^{-2}\right)\, dt$$

(NOTE: Now the DE is also exact).

The solution is: $-y + 2\ln|y| = -\left(t + \ln|t| - \frac{1}{t} + C\right)$

15. $2xy^2 + 2y + (2x^2y + 2x)y' = 0$

Exact. $x^2y^2 + 2xy = C$.

16. $x^3 \dfrac{dy}{dx} = 1 - 2x^2 y$.

Linear: $y' + \frac{2}{x}y = x^{-3}$, with integrating factor $x^2$:

$$y = \frac{\ln|x| + C}{x^2}$$

17. This is separable:

$$\int \frac{1}{1+y^2}\, dy = \int 2 + 2x\, dx \quad \Rightarrow \quad \tan^{-1}(y) = 2x + x^2 + C$$

Solve for $C$: $0 = 0 + C$, so the solution is:

$$y = \tan\left(2x + x^2\right)$$

If we wanted to continue finding the interval on which the solution is valid (not asked for), we could by solving for $x$:

$$x^2 + 2x = \frac{\pi}{2} \qquad x^2 + 2x = -\frac{\pi}{2}$$

The solutions to the first are approx 0.6033 and $-2.6033$. The solutions to the second are not real- Therefore, the interval is $(-2.6033, 0.6033)$. Also for fun, we include the plot of the solution below, with the direction field.

## Misc.

1. Construct a linear first order differential equation whose general solution is given by:

$$y(t) = t - 3 + \frac{C}{t^2}$$

Construct $y'$. The idea will be to produce a linear DE. Therefore, we need to construct $y'$ and compare it to $y$:

$$y' = 1 - 2Ct^{-3}$$

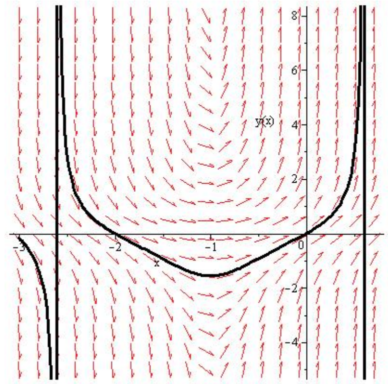


Figure 1: Direction field and solution curve to $y' = 2(1+x)(1+y^2)$. Note the vertical asymptotes.

Add this to some multiple ($t$'s are allowed) of $y$ to get of the arbitrary constant. In this case,

$$y' + \frac{2}{t}y = (1 - 2Ct^{-3}) + 2 - \frac{6}{t} + 2Ct^{-3} = 3 - \frac{6}{t}$$

or,

$$ty' + 2y = 3t - 6$$

2. Construct a linear first order differential equation whose general solution is given by:

$$y(t) = 2\sin(3t) + C\mathrm{e}^{-2t}$$

$$y' = 6\cos(3t) - 2C\mathrm{e}^{-2t}$$

so that: $y' + 2y = 4\sin(3t) + 6\cos(3t)$.

3. Construct an autonomous differential equation that has stable equilibria at $y(t) = 1$ and $y(t) = 3$, and one unstable equilibrium at $y(t) = 2$. (Hint: Draw the phase plot first).

The formula would be something like:

$$y' = -\alpha(y-1)(y-2)(y-3)$$

with $\alpha > 0$.

4. Suppose we have a tank that contains M gallons of water, in which there is $Q_0$ pounds of salt. Liquid is pouring into the tank at a concentration of $r$ pounds per gallon, and at a rate of $\gamma$ gallons per minute. The well mixed solution leaves the tank at a rate of $\gamma$ gallons per minute.

Write the initial value problem that describes the amount of salt in the tank at time $t$, and solve:

$$\frac{dQ}{dt} = r\gamma - \frac{\gamma}{M}Q, \qquad Q(0) = Q_0$$

The solution is:

$$Q = rM + (Q_0 - rM)\mathrm{e}^{-(\gamma/M)t}$$

5. Referring to the previous problem, if let let the system run infinitely long, how much salt will be in the tank? Does it depend on $Q_0$? Does this make sense?

Note that the differential equation for $Q$ is autonomous, so we could do a phase plot (line with a negative slope). Or, we can just take the limit as $t \to \infty$ and see that $Q \to rM$. This does not necessarily depend on $Q_0$; if $Q_0$ starts at equilibrium, $rM$, then $Q$ is constant.

It does make sense. The incoming concentration of salt is $r$ pounds per gallon, so we would expect the long term concentration to be the same, $rM/M = r$.

6. Modify problem 5 if: $M = 100$ gallons, $r = 2$ and the input rate is 2 gallons per minute, and the output rate is 3 gallons per minute. Solve the initial value problem, if $Q_0 = 50$.

$$\frac{dQ}{dt} = 4 - \frac{3}{100+t}Q \qquad Q(0) = 50$$

This goes from being autonomous to linear. In this case, use an integrating factor,

$$e^{\int p(t)\,dt} = e^{3\int \frac{1}{100+t}\,dt} = e^{3\ln|100+t|} = (100+t)^3 \qquad t > -100$$

Continuing, we get:

$$Q(t) = -\frac{50,000,000}{(100+t)^3} + 100 + t$$

7. Suppose an object with mass of 1 kg is dropped from some initial height. Given that the force due to gravity is 9.8 meters per second squared, and assuming a force due to air resistance of $\frac{1}{2}v$, find the initial value problem (and solve it) for the velocity at time $t$.

The general model is: $mv' = mg - kv$. In this case, $m = 1$, $g = 9.8$ and $k = 1/2$. Therefore,

$$v' = 9.8 - \frac{1}{2}v$$

Which is linear (and autonomous). Since the object is being dropped, the initial velocity is zero.

Solve it:

$$v(t) = 19.6\left(1 - e^{-(1/2)t}\right)$$

8. (Continuing with the last problem): At $t = 10$ minutes, the force due to air resistance suddenly changes to $10v$. Model the velocity for $t \geq 10$ (set up and solve the IVP):

The dynamics are now:

$$v' = 9.8 - 10v$$

In order to make $v$ continuous, the initial condition used here will be where the velocity left off after the last problem.

If we make time re-start at zero (so that $t$ is minutes after the previous 10), we would make $v(0) = 19.6\left(1 - e^{-5}\right)$, which is approximately 19.467. The solution for $t$ minutes after the original 10 minutes is:

$$v(t) = 0.98 + 18.487e^{-10t}$$

NOTE: If you did not restart time, the initial condition would be the same, except $v(10) \approx 19.467$, and the solution would be scaled:

$$v(t) = 0.98 + (18.487 \times e^{100})e^{-10t}$$

valid for $t > 10$.

9. (Continuing with the falling object): In a direction field, draw a sketch of the solution. HINT: These are autonomous differential equations, so you should draw the phase plots first!

First get the equilibrium solutions:

$$v = 2 \cdot 9.8 = 19.6 \qquad v = \frac{9.8}{10} = 0.98$$

Now the first phase plot is a line with negative slope through the $y-$ axis (horizontal axis) at 19.6 (a stable equilibrium).

At $t = 10$, the dynamics change (note that on the direction field), and we have a line with negative slope (a stable equilibrium) at 0.98-

Therefore, initially the velocity moves towards equilibrium at 19.8- When the dynamics change, the velocity will now move towards the new equilibrium at 0.98.

Your direction field should look something like Figure 2.

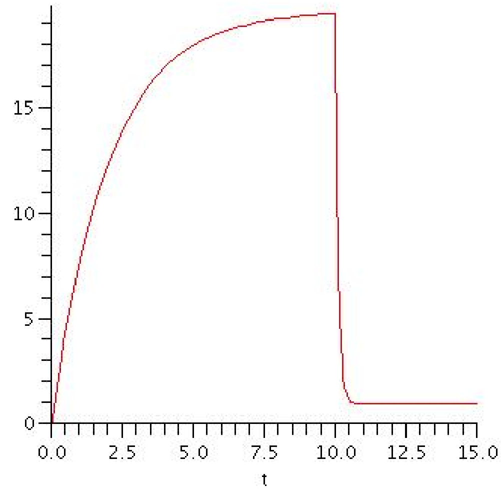


Figure 2: A graph of the velocity for the falling object problem. Note where the equilibria are: $v = 19.6$ for $t < 10$ and $v = 0.98$ for $t > 10$.

10. (Very similar to the quiz and the "Extra Practice")

    (a) Write the autonomous differential equation for modeling population in an environment with a "carrying capacity" of $k$ people.

    $$y' = \alpha y(k - y) \qquad \text{where } \alpha, k > 0$$

    (b) Sketch the phase diagram.

    The phase diagram is a parabola opening downwards, with horizontal intercepts at $y = 0$ and $y = k$. The midpoint is $y = k/2$, since a parabola is symmetric about its vertex.

    (c) Find and classify the equilibrium.

    You should find that $y = 0$ is unstable, and $y = k$ is stable.

    (d) Draw a sketch of $y$ on the direction field, paying particular attention to where $y$ is increasing/decreasing and concave up/down.

    See the quiz solutions and the solutions to the Extra Practice.

    (e) Find the analytic (general) solution. Solve the equation for $y$ (that is, do not leave your answer in implicit form).

    It is separable- Integrate the expression in $y$ using partial fractions:

    $$-\frac{1}{k}\ln|k - y| + \frac{1}{k}\ln|y| = \alpha t + C \quad \Rightarrow \quad \ln\left(\frac{y}{k - y}\right) = \alpha k t + C_2 \quad \Rightarrow \quad \frac{y}{k - y} = Ae^{\alpha k t}$$

    $$\Rightarrow \quad y(t) = \frac{k}{1 + Be^{-\alpha k t}}$$