Summary- Elements of Chapters 7

The goal of this chapter is to solve a linear system of differential equations (we will also be able to solve some special nonlinear systems).

Special Nonlinear Systems

Given the general nonlinear system, $\frac{dx}{dt} = f(x, y)$ and $\frac{dy}{dt} = g(x, y)$, we can find two kinds of solutions: Equilibrium solutions (more in Chapter 9), and solution curves (also known as integral curves) that are solutions to:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g(x, y)}{f(x, y)}$$

If we're lucky, this will simplify to a form from Chapter 2 (first order equations). Notice that with a linear system, $x' = ax + by$, $y' = cx + dy$, then

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by} = \frac{c + d\frac{y}{x}}{a + b\frac{y}{x}}$$

is at least homogeneous (but may also be something else).

Linear Systems

We're talking about three ways to solve a linear system:

- Using dy/dx, as in the last section.
- Converting the system to a second order equation (then use Chapter 3 methods)
- Using eigenvalues and eigenvectors. This last form will also be used to do some analysis in Chapter 9.

We started with some basic matrix algebra- Be sure you know how to perform matrix-vector multiplication and matrix-matrix multiplication for 2×2 matrices.

Eigenvalues and Eigenvectors

1. Definition: Given an $n \times n$ matrix A, if there is a constant λ and a non-zero vector $\mathbf{v}$ so that
$$A\mathbf{v} = \lambda\mathbf{v}$$
then λ is an eigenvalue, and $\mathbf{v}$ is an associated eigenvector.
2. Eigenvectors are not unique. That is, if $\mathbf{v}$ is an eigenvector for A, so is $k\mathbf{v}$ (prove it!).

3. If you're starting to compute them for the first time, start with the original definition and work through to the system:

$$A\mathbf{v} = \lambda\mathbf{v} \Leftrightarrow \begin{array}{rrl} av_1 & +bv_2 & = \lambda v_1 \\ cv_1 & +dv_2 & = \lambda v_2 \end{array} \Leftrightarrow \begin{array}{rrl} (a-\lambda)v_1 & +bv_2 & = 0 \\ cv_1 & +(d-\lambda)v_2 & = 0 \end{array} \tag{1}$$

This system has a non-trivial solution for v_1, v_2 only if the determinant of coefficients is 0:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

And this is the **characteristic equation**. We solve this for the eigenvalues:

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0 \quad \Leftrightarrow \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

where $\mathrm{Tr}(A)$ is the trace of A (which we defined as $a+d$). For each λ, we must go back and solve Equation (1).

4. A note about notation: Often it is easier to use the notation $A - \lambda I$ to represent the matrix:

$$A - \lambda I = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}$$

   - Using this notation, the characteristic equation becomes: $|A - \lambda I| = 0$.
   - Using this notation, the eigenvector equation is: $(A - \lambda I)\mathbf{v} = \mathbf{0}$
   - The generalized eigenvector $\mathbf{w}$ solves: $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Solve $\mathbf{x}' = A\mathbf{x}$ using Eigenvectors/Eigenvalues

We make the ansatz:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v} = \mathrm{e}^{\lambda t} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{\lambda t} v_1 \\ \mathrm{e}^{\lambda t} v_2 \end{bmatrix}$$

We showed that this implies $\lambda, \mathbf{v}$ must be an eigenvalue, eigenvector of the matrix A.

The eigenvalues are found by solving the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

The solution is one of three cases, depending on Δ:

- Real λ_1, λ_2 with two eigenvectors, $\mathbf{v}_1, \mathbf{v}_2$:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$$

- Complex $\lambda = a + ib$, $\mathbf{v}$ (we only need one):

$$\mathbf{x}(t) = C_1 \mathrm{Re}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right) + C_2 \mathrm{Im}\left(\mathrm{e}^{\lambda t}\mathbf{v}\right)$$

  Computational Note: As in Chapter 3, our solutions here are real solutions- That means you should not have an i in your final answer.

- One eigenvalue, one eigenvector $\mathbf{v}$. Get $\mathbf{w}$ that solves $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then:

$$\mathbf{x}(t) = \mathrm{e}^{\lambda t} \left(C_1 \mathbf{v} + C_2 \left(t\mathbf{v} + \mathbf{w} \right) \right)$$

  Computational Note: You should find that there are an infinite number of possible vectors $\mathbf{w}$- Just choose one convenient representative.

You might find this helpful- Below there is a chart comparing the solutions from Chapter 3 to the solutions in Chapter 7:

Chapter 3	Chapter 7
$ay'' + by' + cy = 0$	$\mathbf{x}'(t) = A\mathbf{x}(t)$
Ansatz: $y = \mathrm{e}^{rt}$	Ansatz: $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$
Characteristic Equation: $ar^2 + br + c = 0$	Characteristic Equation: $\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$
3 cases $\Delta > 0$: $y(t) = C_1 \mathrm{e}^{r_1 t} + C_2 \mathrm{e}^{r_2 t}$ $\Delta < 0$, $r = \alpha + \beta i$ $y(t) = C_1 \mathrm{Re}(\mathrm{e}^{rt}) + C_2 \mathrm{Im}(\mathrm{e}^{rt})$ $\Delta = 0$: $y(t) = \mathrm{e}^{rt}(C_1 + C_2 t)$	3 cases $\Delta > 0$: $\mathbf{x}(t) = C_1 \mathrm{e}^{\lambda_1 t}\mathbf{v}_1 + C_2 \mathrm{e}^{\lambda_2 t}\mathbf{v}_2$ $\Delta < 0$, $\lambda = \alpha + \beta i$ $\mathbf{x}(t) = C_1 \mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v}) + C_2 \mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v})$ $\Delta = 0$: $\mathbf{v}$ solves $(A - \lambda I)\mathbf{v} = \mathbf{0}$ (as usual) $\mathbf{w}$ solves $(A - \lambda I)\mathbf{w} = \mathbf{v}$ $\mathbf{x}(t) = \mathrm{e}^{\lambda t} \left(C_1 \mathbf{v} + C_2 \left(t\mathbf{v} + \mathbf{w} \right) \right)$

Classification of the Equilibria

The origin is always an equilibrium solution to $\mathbf{x}' = A\mathbf{x}$, and we can use the Poincaré Diagram to help us classify the origin. The Poincaré Diagram is based on the discriminant:

$$\Delta = (\mathrm{Tr}(A))^2 - 4\det(A)$$

If $\Delta = 0$, we have a parabola in the $(\mathrm{Tr}(A), \det(A))$ plane. Inside the parabola is where $\Delta < 0$ and outside the parabola is where $\Delta > 0$.