Elements of Chapter 9: Nonlinear Systems

To solve $\mathbf{x}' = A\mathbf{x}$, we use the ansatz that $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$. We found that $\lambda$ is an eigenvalue of $A$, and $\mathbf{v}$ an associated eigenvector. We can also summarize the geometric behavior of the solutions by looking at a plot. However, there is an easier way to classify the stability of the origin (as an equilibrium),

To find the eigenvalues, we compute the characteristic equation:

$$
\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0 \quad \lambda = \frac{\text{Tr}(A) \pm \sqrt{\Delta}}{2}
$$

which depends on the discriminant $\Delta$:

- $\Delta > 0$: Real $\lambda_1, \lambda_2$.
- $\Delta < 0$: Complex $\lambda = a + ib$
- $\Delta = 0$: One eigenvalue.

The type of solution depends on $\Delta$, and in particular, where $\Delta = 0$:

$$
\Delta = 0 \quad \Rightarrow \quad 0 = (\text{Tr}(A))^2 - 4\det(A)
$$

This is a parabola in the $(\text{Tr}(A), \det(A))$ coordinate system, inside the parabola is where $\Delta < 0$ (complex roots), and outside the parabola is where $\Delta > 0$. We can then locate the position of our particular trace and determinant using the Poincaré Diagram and it will tell us what the stability will be.

Examples

Given the system where $\mathbf{x}' = A\mathbf{x}$ for each matrix $A$ below, classify the origin using the Poincaré Diagram:

1. $\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$

   SOLUTION: Compute the trace, determinant and discriminant:

   $$
   \text{Tr}(A) = -6 \quad \det(A) = -7 + 16 = 9 \quad \Delta = 36 - 4 \cdot 9 = 0
   $$

   Therefore, we have a “degenerate sink” at the origin.

2. $\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

   SOLUTION: Compute the trace, determinant and discriminant:

   $$
   \text{Tr}(A) = 0 \quad \det(A) = -1 + 10 = 9 \quad \Delta = 0^2 - 4 \cdot 9 = -36
   $$

   The origin is a center.
3. Given the system \( \mathbf{x}' = A\mathbf{x} \) where the matrix \( A \) depends on \( \alpha \), describe how the equilibrium solution changes depending on \( \alpha \) (use the Poincaré Diagram):

(a) \[
\begin{bmatrix}
2 & -5 \\
\alpha & -2
\end{bmatrix}
\]

SOLUTION: The trace is 0, so that puts us on the “det(A)” axis. The determinant is \(-4 + 5\alpha\). If this is positive, we have a center:

\[-4 + 5\alpha > 0 \quad \Rightarrow \quad \alpha > \frac{4}{5} \quad \Rightarrow \quad \text{The origin is a CENTER}\]

\[\alpha < \frac{4}{5} \quad \Rightarrow \quad \text{The origin is a SADDLE}\]

If \( \alpha = \frac{4}{5} \), we have “uniform motion”. That is, \( x_1(t) \) and \( x_2(t) \) will be linear in \( t \) (see if you can find the general solution!).

(b) \[
\begin{bmatrix}
\alpha & 1 \\
-1 & \alpha
\end{bmatrix}
\]

SOLUTION: The trace is \( 2\alpha \) and the discriminant is \( \alpha^2 + 1 \). The discriminant is:

\[\Delta = 4\alpha^2 - 4(\alpha^2 + 1) = 4\alpha^2 - 4\alpha^2 - 4 = -4\]

Therefore, we are always inside the parabola in the upper part of the graph, so the sign of the trace will tell us if we have a SPIRAL SINK \( (\alpha < 0) \), a CENTER \( (\alpha = 0) \), or a SPIRAL SOURCE \( (\alpha > 0) \).

4. In addition to classifying the origin, find the general solution to the system \( \mathbf{x}' = A\mathbf{x} \) using eigenvalues and eigenvectors for the matrix \( A \) below.

\[ A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \]

SOLUTION: The trace is \(-2\) and the determinant is \(5\), and the discriminant is \(4 - 4 \cdot 5 = -16\), so the origin is a SPIRAL SINK. The characteristic equation is

\[\lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 1 + 4 = 0 \quad \Rightarrow \quad (\lambda + 1)^2 = -4\]

and \( \lambda = -1 \pm 2i \). For \( \lambda = -1 + 2i \), we find the corresponding eigenvector:

\[
\begin{align*}
(-1 + 1 - 2i)v_1 - 4v_2 &= 0 \\
v_1 + (-1 + 1 - 2i)v_2 &= 0
\end{align*}
\Rightarrow \quad v_1 = 2iv_2 \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}
\]

Now we compute \( e^{\lambda t}\mathbf{v} \):

\[e^{(-1+2i)t} \begin{bmatrix} 2i \\ 1 \end{bmatrix} = e^{-t}(\cos(2t) + i\sin(2t)) \begin{bmatrix} 2i \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} -2\sin(2t) + i2\cos(2t) \\ \cos(2t) + i\sin(t) \end{bmatrix} \]
Now the solution to the differential equation is:
\[ x(t) = C_1 \text{Re}(e^{\lambda t}v) + C_2 \text{Im}(e^{\lambda t}v) \]

The exponential can be factored out to make it simpler to write:
\[ x(t) = e^{-t} \left( C_1 \begin{bmatrix} -2 \sin(t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} 2 \cos(t) \\ \sin(2t) \end{bmatrix} \right) \]

Just for fun, we could solve this last system using Maple. Just type in:
\[
\text{sys_ode} := \text{diff}(x(t),t) = -x(t) - 4*y(t), \quad \text{diff}(y(t),t) = x(t) - y(t); \\
\text{dsolve([[sys_ode],[x,y]]);}
\]

**Linearizing a Nonlinear System**

The following notes are elements from Sections 9.2 and 9.3.

- Suppose we have an autonomous system of equations:
  \[
  \begin{align*}
  x' &= f(x,y) \\
  y' &= g(x,y)
  \end{align*}
  \]

  Then (as before) we define a point \((a, b)\) to be an **equilibrium point** for the system if \(f(a, b) = 0\) AND \(g(a, b) = 0\) (that is, you must solve the system of equations, not one at a time).

- **Example:** Find the equilibria to:
  \[
  \begin{align*}
  x' &= -(x - y)(1 - x - y) \\
  y' &= x(2 + y)
  \end{align*}
  \]

  SOLUTION: From the second equation, either \(x = 0\) or \(y = -2\). Take each case separately.

  - If \(x = 0\), then the first equation becomes \(y(1 - y)\), so \(y = 0\) or \(y = 1\). So far, we have two equilibria:
    \((0, 0)\) \quad \((0, 1)\)

  - Next, if \(y = -2\) in the second equation, then the first equation becomes
    \[-(x + 2)(1 - x + 2) = 0 \Rightarrow x = -2\) or \(x = 3\)

  We now have two more equilibria:
  \((-2, -2)\) \quad \((3, -2)\)
- **Key Idea:** The “interesting” behavior of a dynamical system is organized around its equilibrium solutions.

- To see what this means, here is the graph of the direction field for the example nonlinear system:

- In order to understand this picture, we will need to linearize the differential equation about its equilibrium.

- Let \( x = a, y = b \) be an equilibrium solution to \( x' = f(x, y) \) and \( y' = g(x, y) \). Then the linearization about \((a, b)\) is the system:

\[
\begin{bmatrix}
  u' \\
  v'
\end{bmatrix} =
\begin{bmatrix}
  f_x(a, b) & f_y(a, b) \\
  g_x(a, b) & g_y(a, b)
\end{bmatrix}
\begin{bmatrix}
  u \\
  v
\end{bmatrix}
\]

where \( u = x - a \) and \( y = v - b \). In our analysis, we really only care about this matrix- You may have used it before, it is called the Jacobian matrix.

- Continuing with our previous example, we compute the Jacobian matrix, then we will insert the equilibria one at a time and perform our local analysis. We then try to put together a global picture of what’s happening.

Recall that the system is:

\[
\begin{align*}
x' &= -(x - y)(1 - x - y) = -x + x^2 + y - y^2 \\
y' &= x(2 + y) = 2x + xy
\end{align*}
\]

The Jacobian matrix for our example is:

\[
\begin{bmatrix}
f_x & f_y \\
g_x & g_y
\end{bmatrix} =
\begin{bmatrix}
-1 + 2x & 1 - 2y \\
2 + y & x
\end{bmatrix}
\]
<table>
<thead>
<tr>
<th>Equilibrium</th>
<th>System</th>
<th>Tr(A)</th>
<th>det(A)</th>
<th>Δ</th>
<th>Poincare</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>$\begin{bmatrix} -1 &amp; 1 \ 2 &amp; 0 \end{bmatrix}$</td>
<td>-1</td>
<td>-2</td>
<td></td>
<td>Saddle</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>$\begin{bmatrix} -1 &amp; -1 \ 3 &amp; 0 \end{bmatrix}$</td>
<td>-1</td>
<td>3</td>
<td>-11</td>
<td>Spiral</td>
</tr>
<tr>
<td>(-2, -2)</td>
<td>$\begin{bmatrix} -5 &amp; 5 \ 0 &amp; -2 \end{bmatrix}$</td>
<td>-7</td>
<td>10</td>
<td>9</td>
<td>Sink</td>
</tr>
<tr>
<td>(3, -2)</td>
<td>$\begin{bmatrix} 5 &amp; 5 \ 0 &amp; 3 \end{bmatrix}$</td>
<td>8</td>
<td>15</td>
<td>4</td>
<td>Source</td>
</tr>
</tbody>
</table>

Here’s the picture again:
<table>
<thead>
<tr>
<th>System</th>
<th>Tr(A)</th>
<th>det(A)</th>
<th>Δ</th>
<th>Poincare</th>
<th>λ</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Fill in</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| \[
\begin{bmatrix}
1 & 2 \\
-5 & -1
\end{bmatrix}
\] | 0     | 9      | -36 | 3i       |        | \[
\begin{bmatrix}
2 \\
-1 + 3i
\end{bmatrix}
\] |
| \[
\begin{bmatrix}
1 & -1 \\
1 & 3
\end{bmatrix}
\] | 4     | 4      | 0   | 2,2      |        | \[
\begin{bmatrix}
-1 & 0 \\
1 & 1
\end{bmatrix}
\] |
| \[
\begin{bmatrix}
-\frac{1}{2} & 1 \\
-1 & -\frac{1}{2}
\end{bmatrix}
\] | -1    | 5/4    | -4  | -\frac{1}{2} + i |        | \[
\begin{bmatrix}
1 \\
i
\end{bmatrix}
\] |
| \[
\begin{bmatrix}
-1 & -1 \\
0 & -\frac{1}{4}
\end{bmatrix}
\] | -5/4  | 1/4    | 9/16| -1, -1/4 |        | \[
\begin{bmatrix}
1 & -4 \\
0 & 3
\end{bmatrix}
\] |

Figure 1: From top to bottom, Center, Degenerate Source, Spiral Sink, Sink. This is for class discussion.
Homework: Elements of Chapter 9, Day 1

1. Fill in the following and under “Poincaré” classify the origin. Then, given the eigenvalues/eigenvectors, also write down the general solution to $x' = Ax$. In the case that there is only one eigenvector, the second column of $V$ shows the generalized eigenvector $w$.

<table>
<thead>
<tr>
<th>System</th>
<th>Tr($A$)</th>
<th>det($A$)</th>
<th>$\Delta$</th>
<th>Poincaré</th>
<th>$\lambda$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{bmatrix} 3 &amp; -2 \ 4 &amp; -1 \end{bmatrix}$</td>
<td>1 2i</td>
<td>$1 + 2i$</td>
<td>1 - i</td>
<td>1 + 2i</td>
<td>$\begin{bmatrix} 1 \ 1 - i \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\begin{bmatrix} 2 &amp; -1 \ 3 &amp; -2 \end{bmatrix}$</td>
<td>-1,1</td>
<td>-1,1</td>
<td>1 1 3 1</td>
<td>-i 1 1</td>
<td>$\begin{bmatrix} 1 \ -i \ 1 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\begin{bmatrix} 0 &amp; 2 \ -2 &amp; 0 \end{bmatrix}$</td>
<td>2i</td>
<td>2i</td>
<td>-i</td>
<td>1</td>
<td>$\begin{bmatrix} -i \ 1 \end{bmatrix}$</td>
<td></td>
</tr>
<tr>
<td>$\begin{bmatrix} 4 &amp; -2 \ 8 &amp; -4 \end{bmatrix}$</td>
<td>0,0</td>
<td>0,0</td>
<td>1 0 2 -1/2</td>
<td>0 0</td>
<td>$\begin{bmatrix} 1 \ 2 \ -1/2 \end{bmatrix}$</td>
<td></td>
</tr>
</tbody>
</table>

2. Explain how the classification of the origin changes by changing the $\alpha$ in the system:

(a) $x' = \begin{bmatrix} 0 & \alpha \\ 1 & -2 \end{bmatrix} x$
(b) $x' = \begin{bmatrix} 2 & \alpha \\ 1 & -1 \end{bmatrix} x$
(c) $x' = \begin{bmatrix} \alpha & 10 \\ -1 & -4 \end{bmatrix} x$

Hint: Use a number line to keep track of where the trace, determinant and discriminant change sign.

3. For the following nonlinear systems, find the equilibrium solutions (the derivatives are with respect to $t$, as usual).

(a) $x' = x - xy$, $y' = y + 2xy$
(b) $x' = y(2 - x - y)$, $y' = -x - y - 2xy$
(c) $x' = 1 + 2y$, $y' = 1 - 3x^2$

4. For each of the systems in the previous problem, find the Jacobian matrix, then linearize about each equilibrium. Use the Poincaré Diagram to classify each equilibrium solution.