Elements of Chapter 9: Nonlinear Systems

To solve $\mathbf{x}' = A\mathbf{x}$, we use the ansatz that $\mathbf{x}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$. We found that λ is an eigenvalue of A, and $\mathbf{v}$ an associated eigenvector. We can also summarize the geometric behavior of the solutions by looking at a plot- However, there is an easier way to classify the stability of the origin (as an equilibrium),

To find the eigenvalues, we compute the characteristic equation:

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0 \qquad \lambda = \frac{\mathrm{Tr}(A) \pm \sqrt{\Delta}}{2}$$

which depends on the discriminant Δ:

- $\Delta > 0$: Real λ_1, λ_2.
- $\Delta < 0$: Complex $\lambda = a + ib$
- $\Delta = 0$: One eigenvalue.

The type of solution depends on Δ, and in particular, where $\Delta = 0$:

$$\Delta = 0 \quad \Rightarrow \quad 0 = (\mathrm{Tr}(A))^2 - 4\det(A)$$

This is a parabola in the $(\mathrm{Tr}(A), \det(A))$ coordinate system, inside the parabola is where $\Delta < 0$ (complex roots), and outside the parabola is where $\Delta > 0$. We can then locate the position of our particular trace and determinant using the Poincaré Diagram and it will tell us what the stability will be.

Examples

Given the system where $\mathbf{x}' = A\mathbf{x}$ for each matrix A below, classify the origin using the Poincaré Diagram:

1. $\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$

   SOLUTION: Compute the trace, determinant and discriminant:

   $$\mathrm{Tr}(A) = -6 \qquad \mathrm{Det}(A) = -7 + 16 = 9 \qquad \Delta = 36 - 4 \cdot 9 = 0$$

   Therefore, we have a "degenerate sink" at the origin.

2. $\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

   SOLUTION: Compute the trace, determinant and discriminant:

   $$\mathrm{Tr}(A) = 0 \qquad \mathrm{Det}(A) = -1 + 10 = 9 \qquad \Delta = 0^2 - 4 \cdot 9 = -36$$

   The origin is a **center**.

3. Given the system $\mathbf{x}' = A\mathbf{x}$ where the matrix A depends on α, describe how the equilibrium solution changes depending on α (use the Poincaré Diagram):

(a) $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$

SOLUTION: The trace is 0, so that puts us on the "det(A)" axis. The determinant is $-4 + 5\alpha$. If this is positive, we have a center:

$$-4 + 5\alpha > 0 \quad \Rightarrow \quad \alpha > \frac{4}{5} \quad \Rightarrow \quad \text{The origin is a CENTER}$$

$$\alpha < \frac{4}{5} \quad \Rightarrow \quad \text{The origin is a SADDLE}$$

If $\alpha = \frac{4}{5}$, we have "uniform motion". That is, $x_1(t)$ and $x_2(t)$ will be linear in t (see if you can find the general solution!).

(b) $\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$

SOLUTION: The trace is 2α and the discriminant is $\alpha^2 + 1$. The discriminant is:

$$\Delta = 4\alpha^2 - 4(\alpha^2 + 1) = 4\alpha^2 - 4\alpha^2 - 4 = -4$$

Therefore, we are always inside the parabola in the upper part of the graph, so the sign of the trace will tell us if we have a SPIRAL SINK ($\alpha < 0$), a CENTER ($\alpha = 0$), or a SPIRAL SOURCE ($\alpha > 0$).

4. In addition to classifying the origin, find the general solution to the system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors for the matrix A below.

$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$

SOLUTION: The trace is -2 and the determinant is 5, and the discriminant is $4 - 4 \cdot 5 = -16$, so the origin is a SPIRAL SINK. The characteristic equation is

$$\lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 1 + 4 = 0 \quad \Rightarrow \quad (\lambda + 1)^2 = -4$$

and $\lambda = -1 \pm 2i$. For $\lambda = -1 + 2i$, we find the corresponding eigenvector:

$$\begin{array}{rl} (-1 + 1 - 2i)v_1 - 4v_2 & = 0 \\ v_1 + (-1 + 1 - 2i)v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} v_1 & = 2iv_2 \\ v_2 & = v_2 \end{array} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

Now we compute $\mathrm{e}^{\lambda t}\mathbf{v}$:

$$\mathrm{e}^{(-1+2i)t} \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \mathrm{e}^{-t}(\cos(2t) + i\sin(2t)) \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \mathrm{e}^{-t} \begin{bmatrix} -2\sin(2t) + i2\cos(2t) \\ \cos(2t) + i\sin(t) \end{bmatrix}$$

Now the solution to the differential equation is:

$$\mathbf{x}(t) = C_1 \mathrm{Re}(\mathrm{e}^{\lambda t}\mathbf{v}) + C_2 \mathrm{Im}(\mathrm{e}^{\lambda t}\mathbf{v})$$

The exponential can be factored out to make it simpler to write:

$$\mathbf{x}(t) = \mathrm{e}^{-t}\left(C_1 \begin{bmatrix} -2\sin(t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} 2\cos(t) \\ \sin(2t) \end{bmatrix} \right)$$

Just for fun, we could solve this last system using Maple. Just type in:

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sys_ode := diff(x(t),t)= -x(t)-4*y(t),  diff(y(t),t)= x(t)-y(t);
dsolve([sys_ode],[x,y]);
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Linearizing a Nonlinear System

The following notes are elements from Sections 9.2 and 9.3.

- Suppose we have an autonomous system of equations:

  $$\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}$$

  Then (as before) we define a point (a, b) to be an **equilibrium point** for the system if $f(a, b) = 0$ AND $g(a, b) = 0$ (that is, you must solve the system of equations, not one at a time).

- **Example:** Find the equilibria to:

  $$\begin{aligned} x' &= -(x-y)(1-x-y) \\ y' &= x(2+y) \end{aligned}$$

  SOLUTION: From the second equation, either $x = 0$ or $y = -2$. Take each case separately.

  - If $x = 0$, then the first equation becomes $y(1 - y)$, so $y = 0$ or $y = 1$. So far, we have two equilibria:

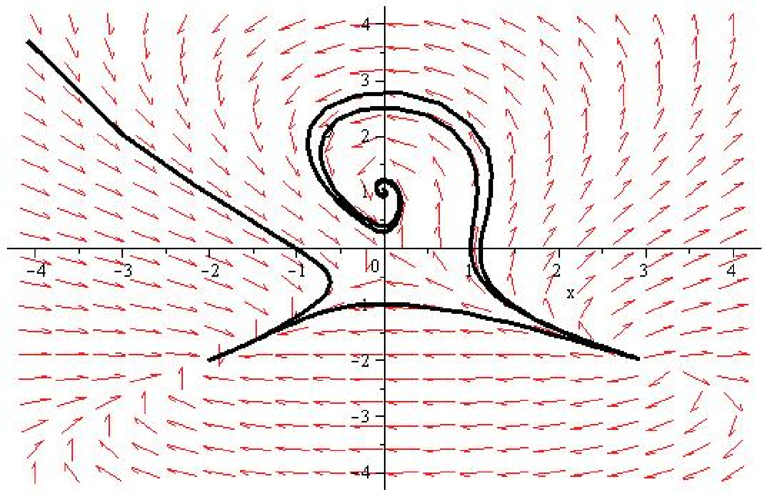
    $$(0,0) \qquad (0,1)$$

  - Next, if $y = -2$ in the second equation, then the first equation becomes

    $$-(x+2)(1-x+2) = 0 \quad \Rightarrow \quad x = -2 \text{ or } x = 3$$

    We now have two more equilibria:

    $$(-2,-2) \qquad (3,-2)$$



- **Key Idea:** The "interesting" behavior of a dynamical system is organized around its equilibrium solutions.

- To see what this means, here is the graph of the direction field for the example nonlinear system:

- In order to understand this picture, we will need to linearize the differential equation about its equilibrium.

- Let $x = a, y = b$ be an equilibrium solution to $x' = f(x, y)$ and $y' = g(x, y)$. Then the linearization about (a, b) is the system:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} f_x(a,b) & f_y(a,b) \\ g_x(a,b) & g_y(a,b) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

  where $u = x - a$ and $y = v - b$. In our analysis, we really only care about this matrix- You may have used it before, it is called the Jacobian matrix.

- Continuing with our previous example, we compute the Jacobian matrix, then we will insert the equilibria one at a time and perform our local analysis. We then try to put together a global picture of what's happening.

  Recall that the system is:

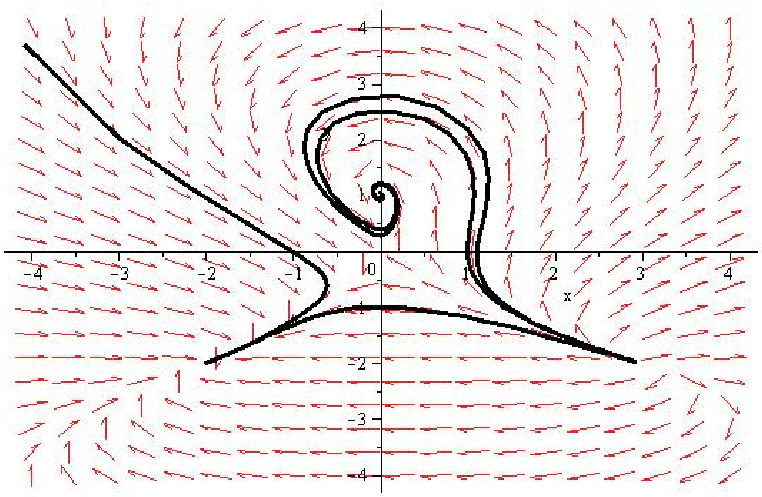
$$\begin{array}{lll} x' & = -(x-y)(1-x-y) & = -x + x^2 + y - y^2 \\ y' & = x(2+y) & = 2x + xy \end{array}$$

  The Jacobian matrix for our example is:

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -1+2x & 1-2y \\ 2+y & x \end{bmatrix}$$

Equilibrium	System	Tr(A)	det(A)	Δ	Poincare
$(0,0)$	$\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$	-1	-2		Saddle
$(0,1)$	$\begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$	-1	3	-11	Spiral Sink
$(-2,-2)$	$\begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$	-7	10	9	Sink
$(3,-2)$	$\begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$	8	15	4	Source

Here's the picture again:



System	Tr(A)	det(A)	Δ	Poincare Fill in	λ	V
$\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$	0	9	-36		$3i$	$\begin{bmatrix} 2 \\ -1+3i \end{bmatrix}$
$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$	4	4	0		$2, 2$	$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$
$\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$	-1	$5/4$	-4		$-\frac{1}{2}+i$	$\begin{bmatrix} 1 \\ i \end{bmatrix}$
$\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$	$-5/4$	$1/4$	$9/16$		$-1, -1/4$	$\begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix}$

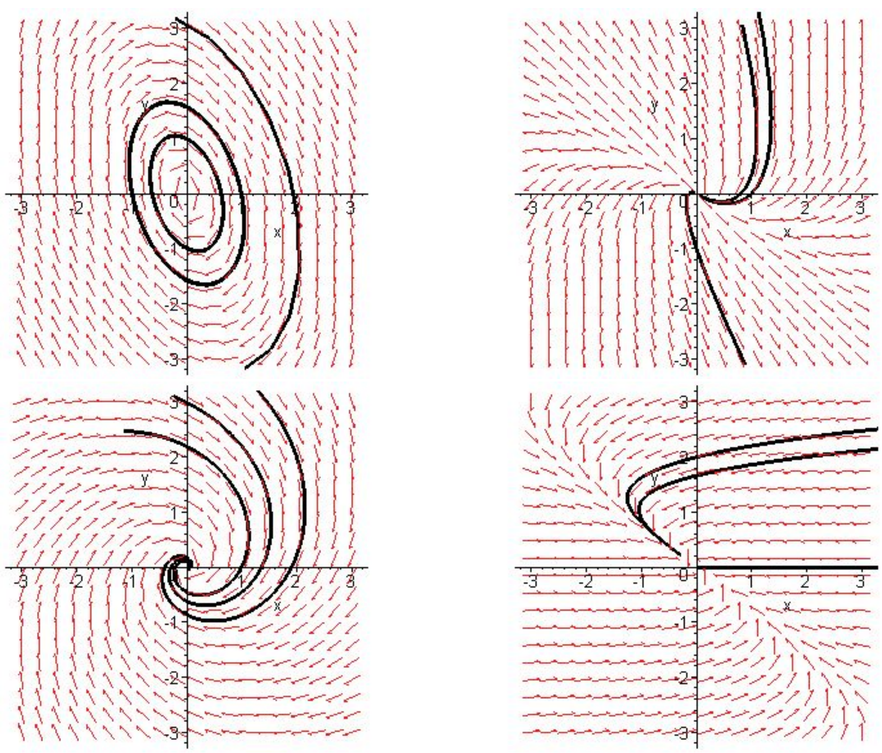


Figure 1: From top to bottom, Center, Degenerate Source, Spiral Sink, Sink. **This is for class discussion**.

Homework: Elements of Chapter 9, Day 1

1. Fill in the following and under "Poincaré" classify the origin. Then, given the eigenvalues/eigenvectors, also write down the general solution to $\mathbf{x}' = A\mathbf{x}$. In the case that there is only one eigenvector, the second column of V shows the generalized eigenvector $\mathbf{w}$.

System	Tr(A)	det(A)	Δ	Poincare Fill in	λ	V
$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$					$1+2i$	$\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$
$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$					$-1, 1$	$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$					$2i$	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$					$0, 0$	$\begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix}$

2. Explain how the classification of the origin changes by changing the α in the system:

   (a) $\mathbf{x}' = \begin{bmatrix} 0 & \alpha \\ 1 & -2 \end{bmatrix} \mathbf{x}$

   (b) $\mathbf{x}' = \begin{bmatrix} 2 & \alpha \\ 1 & -1 \end{bmatrix} \mathbf{x}$

   (c) $\mathbf{x}' = \begin{bmatrix} \alpha & 10 \\ -1 & -4 \end{bmatrix} \mathbf{x}$

   Hint: Use a number line to keep track of where the trace, determinant and discriminant change sign.

3. For the following *nonlinear* systems, find the equilibrium solutions (the derivatives are with respect to t, as usual).

   (a) $x' = x - xy$, $y' = y + 2xy$

   (b) $x' = y(2 - x - y)$, $y' = -x - y - 2xy$

   (c) $x' = 1 + 2y$, $y' = 1 - 3x^2$

4. For each of the systems in the previous problem, find the Jacobian matrix, then linearize about each equilibrium. Use the Poincaré Diagram to classify each equilibrium solution.