

## Corrected Examples (to Notes for Ch 9)

Given the system where  $\mathbf{x}' = A\mathbf{x}$  for each matrix  $A$  below, classify the origin using the Poincaré Diagram:

1.  $\begin{bmatrix} 1 & -4 \\ 4 & -7 \end{bmatrix}$

SOLUTION: Compute the trace, determinant and discriminant:

$$\text{Tr}(A) = -6 \quad \text{Det}(A) = -7 + 16 = 9 \quad \Delta = 36 - 4 \cdot 9 = 0$$

Therefore, we have a “degenerate sink” at the origin.

2.  $\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

SOLUTION: Compute the trace, determinant and discriminant:

$$\text{Tr}(A) = 0 \quad \text{Det}(A) = -1 + 10 = 9 \quad \Delta = 0^2 - 4 \cdot 9 = -36$$

The origin is a **center**.

3. Given the system  $\mathbf{x}' = A\mathbf{x}$  where the matrix  $A$  depends on  $\alpha$ , describe how the equilibrium solution changes depending on  $\alpha$  (use the Poincaré Diagram):

(a)  $\begin{bmatrix} 2 & -5 \\ \alpha & -2 \end{bmatrix}$

SOLUTION: The trace is 0, so that puts us on the “det(A)” axis.

The determinant is  $-4 + 5\alpha$ . If this is positive, we have a center:

$$-4 + 5\alpha > 0 \quad \Rightarrow \quad \alpha > \frac{4}{5} \quad \Rightarrow \quad \text{The origin is a CENTER}$$

$$\alpha < \frac{4}{5} \quad \Rightarrow \quad \text{The origin is a SADDLE}$$

If  $\alpha = \frac{4}{5}$ , we have “uniform motion”. That is,  $x_1(t)$  and  $x_2(t)$  will be linear in  $t$  (see if you can find the general solution!).

(b)  $\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$

SOLUTION: The trace is  $2\alpha$  and the discriminant is  $\alpha^2 + 1$ . The discriminant is:

$$\Delta = 4\alpha^2 - 4(\alpha^2 + 1) = 4\alpha^2 - 4\alpha^2 - 4 = -4$$

Therefore, we are always inside the parabola in the upper part of the graph, so the sign of the trace will tell us if we have a SPIRAL SINK ( $\alpha < 0$ ), a CENTER ( $\alpha = 0$ ), or a SPIRAL SOURCE ( $\alpha > 0$ ).

4. In addition to classifying the origin, find the general solution to the system  $\mathbf{x}' = A\mathbf{x}$  using eigenvalues and eigenvectors for the matrix  $A$  below.

$$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$$

SOLUTION: The trace is  $-2$  and the determinant is  $5$ , and the discriminant is  $4 - 4 \cdot 5 = -16$ , so the origin is a SPIRAL SINK. The characteristic equation is

$$\lambda^2 + 2\lambda + 5 = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 1 + 4 = 0 \quad \Rightarrow \quad (\lambda + 1)^2 = -4$$

and  $\lambda = -1 \pm 2i$ . For  $\lambda = -1 + 2i$ , we find the corresponding eigenvector:

$$\begin{aligned} (-1 + 1 - 2i)v_1 - 4v_2 &= 0 \\ v_1 + (-1 + 1 - 2i)v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} v_1 &= 2iv_2 \\ v_2 &= v_2 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 2i \\ 1 \end{bmatrix}$$

Now we compute  $e^{\lambda t}\mathbf{v}$ :

$$e^{(-1+2i)t} \begin{bmatrix} 2i \\ 1 \end{bmatrix} = e^{-t}(\cos(2t) + i\sin(2t)) \begin{bmatrix} 2i \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} -2\sin(2t) + i2\cos(2t) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

Now the solution to the differential equation is:

$$\mathbf{x}(t) = C_1 \operatorname{Re}(e^{\lambda t}\mathbf{v}) + C_2 \operatorname{Im}(e^{\lambda t}\mathbf{v})$$

The exponential can be factored out to make it simpler to write:

$$\mathbf{x}(t) = e^{-t} \left( C_1 \begin{bmatrix} -2\sin(t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} 2\cos(t) \\ \sin(2t) \end{bmatrix} \right)$$

Just for fun, we could solve this last system using Maple. Just type in:

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sys_ode := diff(x(t),t)= -x(t)-4*y(t), diff(y(t),t)= x(t)-y(t);
dsolve([sys_ode],[x,y]);
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# Linearizing a Nonlinear System

The following notes are elements from Sections 9.2 and 9.3.

- Suppose we have an autonomous system of equations:

$$\begin{aligned}x' &= f(x, y) \\ y' &= g(x, y)\end{aligned}$$

Then (as before) we define a point  $(a, b)$  to be an **equilibrium point** for the system if  $f(a, b) = 0$  AND  $g(a, b) = 0$  (that is, you must solve the system of equations, not one at a time).

- **Example:** Find the equilibria to:

$$\begin{aligned}x' &= -(x - y)(1 - x - y) \\ y' &= x(2 + y)\end{aligned}$$

SOLUTION: From the second equation, either  $x = 0$  or  $y = -2$ . Take each case separately.

- If  $x = 0$ , then the first equation becomes  $y(1 - y)$ , so  $y = 0$  or  $y = 1$ . So far, we have two equilibria:

$$(0, 0) \quad (0, 1)$$

- Next, if  $y = -2$  in the second equation, then the first equation becomes

$$-(x + 2)(1 - x + 2) = 0 \quad \Rightarrow \quad x = -2 \text{ or } x = 3$$

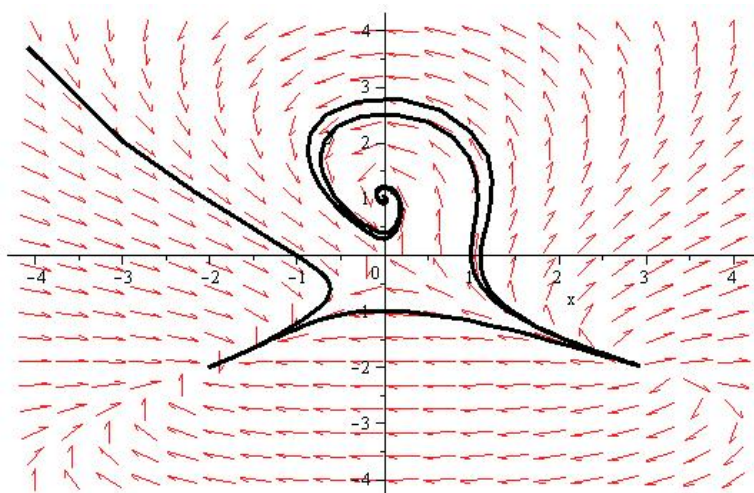
We now have two more equilibria:

$$(-2, -2) \quad (3, -2)$$

- **Key Idea:** The “interesting” behavior of a dynamical system is organized around its equilibrium solutions.
- To see what this means, here is the graph of the direction field for the example nonlinear system:
- In order to understand this picture, we will need to linearize the differential equation about its equilibrium.
- Let  $x = a, y = b$  be an equilibrium solution to  $x' = f(x, y)$  and  $y' = g(x, y)$ . Then the linearization about  $(a, b)$  is the system:

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where  $u = x - a$  and  $y = v - b$ . In our analysis, we really only care about this matrix- You may have used it before, it is called the Jacobian matrix.



- Continuing with our previous example, we compute the Jacobian matrix, then we will insert the equilibria one at a time and perform our local analysis. We then try to put together a global picture of what's happening.

Recall that the system is:

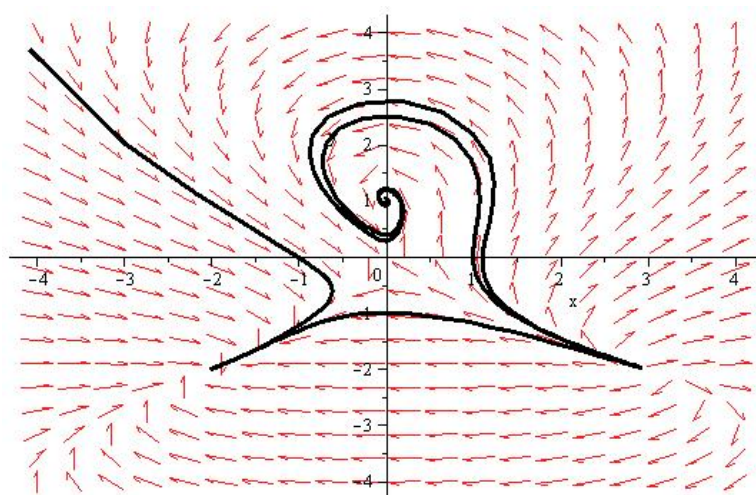
$$\begin{aligned} x' &= -(x-y)(1-x-y) = -x + x^2 + y - y^2 \\ y' &= x(2+y) = 2x + xy \end{aligned}$$

The Jacobian matrix for our example is:

$$\begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} -1+2x & 1-2y \\ 2+y & x \end{bmatrix}$$

Equilibrium	System	Tr(A)	det(A)	$\Delta$	Poincare
$(0, 0)$	$\begin{bmatrix} -1 & 1 \\ 2 & 0 \end{bmatrix}$	-1	-2		Saddle
$(0, 1)$	$\begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$	-1	3	-11	Spiral Sink
$(-2, -2)$	$\begin{bmatrix} -5 & 5 \\ 0 & -2 \end{bmatrix}$	-7	10	9	Sink
$(3, -2)$	$\begin{bmatrix} 5 & 5 \\ 0 & 3 \end{bmatrix}$	8	15	4	Source

Here's the picture again:



System	$\text{Tr}(A)$	$\det(A)$	$\Delta$	Poincare Fill in	$\lambda$	$V$
$\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$	0	9	-36		$3i$	$\begin{bmatrix} 2 \\ -1 + 3i \end{bmatrix}$
$\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$	4	4	0		$2, 2$	$\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$
$\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix}$	-1	5/4	-4		$-\frac{1}{2} + i$	$\begin{bmatrix} 1 \\ i \end{bmatrix}$
$\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$	-5/4	1/4	9/16		$-1, -1/4$	$\begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix}$

## Homework: Elements of Chapter 9, Day 1

- Fill in the following and under “Poincaré” classify the origin. Then, given the eigenvalues/eigenvectors, also write down the general solution to  $\mathbf{x}' = A\mathbf{x}$ . In the case that there is only one eigenvector, the second column of  $V$  shows the generalized eigenvector  $\mathbf{w}$ .

System	$\text{Tr}(A)$	$\det(A)$	$\Delta$	Poincare Fill in	$\lambda$	$V$
$\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$					$1 + 2i$	$\begin{bmatrix} 1 \\ 1 - i \end{bmatrix}$
$\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}$					$-1, 1$	$\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$
$\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$					$2i$	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$
$\begin{bmatrix} 4 & -2 \\ 8 & -4 \end{bmatrix}$					$0, 0$	$\begin{bmatrix} 1 & 0 \\ 2 & -1/2 \end{bmatrix}$

- Explain how the classification of the origin changes by changing the  $\alpha$  in the system:

(a)  $\mathbf{x}' = \begin{bmatrix} 0 & \alpha \\ 1 & -2 \end{bmatrix} \mathbf{x}$

(b)  $\mathbf{x}' = \begin{bmatrix} 2 & \alpha \\ 1 & -1 \end{bmatrix} \mathbf{x}$

(c)  $\mathbf{x}' = \begin{bmatrix} \alpha & 10 \\ -1 & -4 \end{bmatrix} \mathbf{x}$

Hint: Use a number line to keep track of where the trace, determinant and discriminant change sign.

- For the following *nonlinear* systems, find the equilibrium solutions (the derivatives are with respect to  $t$ , as usual).

(a)  $x' = x - xy, y' = y + 2xy$

(b)  $x' = y(2 - x - y), y' = -x - y - 2xy$

(c)  $x' = 1 + 2y, y' = 1 - 3x^2$

- For each of the systems in the previous problem, find the Jacobian matrix, then linearize about each equilibrium. Use the Poincaré Diagram to classify each equilibrium solution.

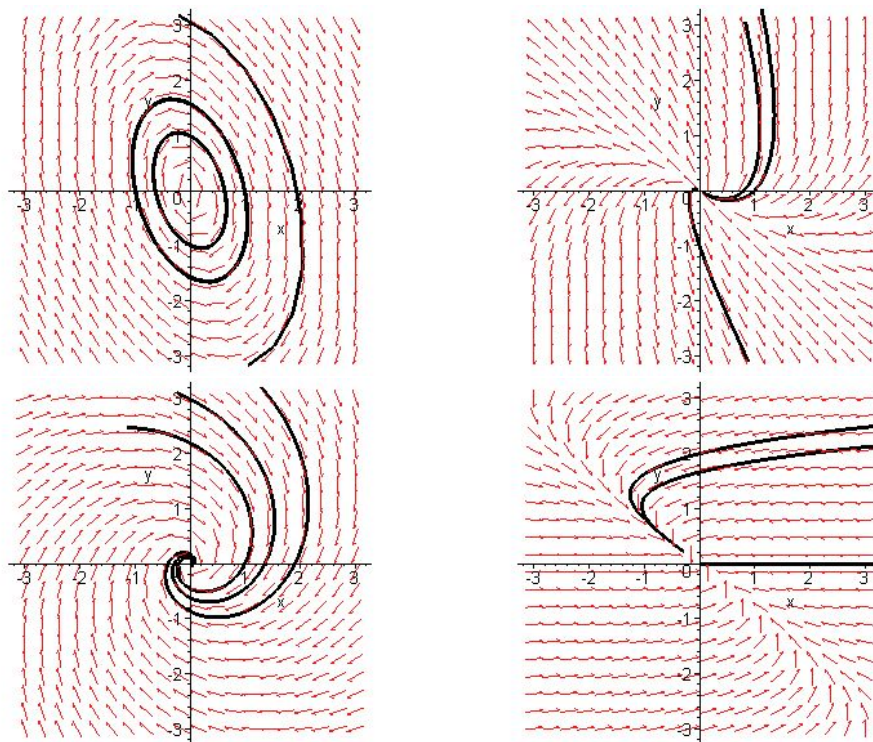


Figure 1: From top to bottom, Center, Degenerate Source, Spiral Sink, Sink.  
**This is for class discussion.**