Summary: Chapter 6, 5.1-5.4

Here is a summary of the material in Chapter 6 and sections 5.1-5.4.

6.1

- Know the definition of the Laplace transform.
- When will the Laplace transform exist? If the function is piecewise continuous and of exponential order. Know these definitions; be able to determine if a function is of exponential order.
- Show that $\mathcal{L}$ is a linear operator.
- Given the definition of $\Gamma(p+1)$, show that $\Gamma(1) = 1$ and, if n is a positive integer, $\Gamma(n+1) = n!$ (Hint: This is not done using the integral, but by showing $\Gamma(n+1) = n\Gamma(n)$).

6.2

- Prove (using the definition of $\mathcal{L}$):

$$\mathcal{L}(f'(t)) = sF(s) - f(0) \qquad \mathcal{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

  where $F(s) = \mathcal{L}(f(t))$.
- Prove table entries: 1-3, 5-6, 9-10, 19.
- Use the table to invert the transform, assuming the inverse is linear (Here is where all the algebra comes in). When should we factor a quadratic in the denominator (versus complete the square)?

6.3-6.4

- Define the Heaviside function $u_c(t)$ and compute its Laplace transform (from the definition).
- Understand how $u_c(t)$ can be used as an "on-off" switch for a differential equation.
- Convert a piecewise defined function into an equivalent function using the Heaviside, and vice versa.
- Use the table entry:

$$\mathcal{L}(u_c(t)f(t-c)) = \mathrm{e}^{-cs}F(s)$$

  In particular, be able to take the transform of something like $u_3(t)\, t^2$.
- Use the table entry:

$$\mathcal{L}(\mathrm{e}^{ct}f(t)) = F(s-c)$$

  For example, use it to show Table Entry 11 from Table Entry 3, and Entries 9 and 10 from 5 and 6.
- The extra wrinkle introduced in 6.4 is to be able to solve the DE's when the forcing function (the function on the right hand side of the DE) uses summation notation.
- Be able to write out the solution as a piecewise defined function (without the Heaviside function).

6.5

- Define the Dirac $\delta-$function (or "unit impulse function"), and be able to compute its Laplace transform. Two important properties:

$$\int_{-\infty}^{\infty} \delta(t-c)\, dt = 1 \qquad \int_{-\infty}^{\infty} \delta(t-c)f(t)\, dt = f(c)$$

  (I won't ask you to prove these).

- Used to model a force of very short duration with finite strength (for example, a hammer strike). We saw that using the Dirac function is like imparting a velocity of +1 on the mass-spring system. That is, the solutions to the following were identical:

$$ay'' + by' + cy = 0 \quad y(0) = 0, y'(0) = 1 \qquad\qquad ay'' + by' + cy = \delta(t), y(0) = 0, y'(0) = 0$$

- Solve DEs that use the Dirac $\delta-$function. In particular, if the forcing function again uses summation notation (like in 6.4).

6.6

- Know the definition of the convolution, and be able to compute it directly for "simple" cases.
- Know "The Convolution Theorem":

$$\mathcal{L}^{-1}(F(s)G(s)) = f(t) * g(t)$$

- Use the Convolution Theorem to compute a convolution (using partial fractions).
- You do not need to know the *impulse response* or *transfer function* that are mentioned in the last example in the text.
- In the exercises, the new type of problem is the *integral equation*. Be able to solve these using the Laplace transform (like 23-28, part (a)).

5.1

- Review of power series.
- The ratio test
- Find the radius of convergence and the interval of convergence.
- Recall the template series:

$$\sum \frac{(-1)^n}{n} \qquad \sum \frac{1}{n} \qquad \sum_{n=k}^{\infty} ar^n = \frac{ar^k}{1-r}, |r| < 1$$

- And the template Maclaurin series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1 \qquad \mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

  And it is useful to recall the series for the sine (with odd powers) and cosine (with even powers) as well.
- Algebra: Be able to manipulate the index of summation (5.1, Examples 3-6)
- Know the underlying theorem for using series to solve DEs:

$$\sum_n c_n x^n = 0 \text{ for all } x \in I \quad \Rightarrow \quad c_n = 0 \text{ for each } n$$

  where I is an open interval (in the interval of convergence).

  This is the extension of a theorem we use all the time in solving partial fraction problems as well: If two polynomials are equal for all x in an open interval I, then their coefficients must be the same.

5.2

- Consider the model equation:
$$P(x)y'' + Q(x)y' + R(x)y = 0$$
where P, Q, and R are polynomials. Then a point x_0 is called an **ordinary point** if $P(x_0) \neq 0$. Alternatively, if $P(x_0) = 0$, then the point is called a **singular point**. In Sections 5.2 and 5.3, we only find series solutions about ordinary points.

- Our ansatz for 5.1-5.3 is that y is analytic at x_0:
$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!}(x - x_0)^n$$

- Be able to substitute the general series into a given DE, to find the **recurrence relation** for the coefficients.

5.3

- Know the Existence and Uniqueness Theorem for Series (Theorem 5.3.1) for
$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad \Rightarrow \qquad y'' + p(x)y' + q(x)y = 0$$

- Be able to compute the series solution (up to a few terms) by directly computing the derivatives $(y''(0), y'''(0), y^{(4)}(0)$, etc (we actually did this on our first day in Chapter 5).

- Can we predict the minimum radius of convergence for our solution? Theorem 5.3.1 tells us that it is at least as large as the minimum of the radii of convergence of the series for p, q.

  How can we find those? The key is that p, q are rational functions (a fraction with polynomials in numerator and denominator). Assume p, q are reduced fractions, and the denominators are not zero at x_0. Then they both are analytic at x_0, and the radius of convergence for each will be the distance from x_0 to the closest zero in the denominator (measured in the complex plane).

- Example: What is minimum radius of convergence for the solution to
$$(1 + x^2)y'' + xy' + 3y = 0 \qquad x_0 = 6$$
SOLUTION: We look at p, q in reduced form:
$$p(x) = \frac{x}{1 + x^2} \qquad q(x) = \frac{3}{1 + x^2}$$
They both have zeros in the denominator at $x = \pm i$. Therefore, the minimum radius will be the shortest distance (in the complex plane) from $x_0 = 6 + 0i$ and $x = 0 \pm i$- You can compute this as if we were computing the distance between points $(6, 0)$ and $(0, \pm 1)$- In this case,
$$\sqrt{(6 - 0)^2 + (0 - 1)^2} = \sqrt{37}$$
ANSWER: The radius of convergence will be at least $\sqrt{37}$.

5.4

- The Euler Equation: $x^2 y'' + \alpha x y' + \beta y = 0$

- Use the ansatz: $y = x^r$, get the polynomial:
$$r(r - 1) + \alpha r + \beta = 0$$
Solve it using the quadratic formula. Three possible outcomes:
  - Real, distinct roots: $y_1 = x^{r_1}$, $y_2 = x^{r_2}$

- Equal roots: $y_1 = x^r$, $y_2 = x^r \ln(x)$
- Complex roots: $r = \lambda \pm \mu i$

$$y_1 = x^\lambda \cos(\mu \ln(x)) \qquad y_2 = x^\lambda \sin(\mu \ln(x))$$

NOTE 1: We already saw this material in Section 3.3 (p 165), where we made the substitution (we switched notation here): $t = \ln(x)$, and transformed the Euler equation to:

$$\ddot{y} + (\alpha - 1)\dot{y} + \beta y = 0$$

Then, the formulas are easy to remember, since:

$$\mathrm{e}^{rt} = \mathrm{e}^{r\ln(x)} = x^r \qquad t\mathrm{e}^{rt} = x^r \ln(x) \qquad \mathrm{e}^{\lambda t}\cos(\mu t) = x^\lambda \cos(\mu \ln(x))$$

- Regular Singular Points: If x_0 is a singular point, then for the DE in the form

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

we say that x_0 is a regular singular point if the following two limits exist:

$$\lim_{x\to x_0} (x - x_0)\frac{Q}{P} = p_0 \qquad \lim_{x\to x_0} (x - x_0)^2\frac{R}{P} = q_0$$

Furthermore, the associated Euler Equation is then given by:

$$(x - x_0)^2 y'' + p_0(x - x_0)y' + q_0 y = 0$$

so that the form of the characteristic equation is: $r(r-1) + p_0 r + q_0 = 0$. These are the "exponents at the singularity". (Note: For the exam, we'll stay at the origin, $x_0 = 0$, when looking at singular points).