Chapter 3, Sect 5

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The Set Up

Find solutions to

$$ay'' + by' + cy = g(t)$$

For certain types of $g(t)$
($g(t)$ is called the forcing function).

If $y_h(t)$ is the general solution to $ay'' + by' + cy = 0$, and:

- $y_{p_1}(t)$ solves $ay'' + by' + cy = g_1(t)$
- $y_{p_2}(t)$ solves $ay'' + by' + cy = g_2(t)$
- and so on...
- $y_{p_n}(t)$ solves $ay'' + by' + cy = g_n(t)$

then the full general solution to:

$$ay'' + by' + cy = g_1(t) + g_2(t) + \cdots + g_n(t)$$

is:

$$y_h(t) + y_{p_1}(t) + y_{p_2}(t) + \cdots + y_{p_n}(t)$$

Proof: From the last slide, let

$$L(y) = ay'' + by' + cy$$

Then L is a linear operator, with $L(y_h) = 0$, $L(y_{p_1}(t)) = g_1(t)$, $L(y_{p_2}(t)) = g_2(t)$, and so on. Therefore,

$$L(y_h + y_{p_1} + y_{p_2} + \cdots + y_{p_n}) = 0 + g_1 + g_2 + \cdots + g_n$$

by the linearity of the operator L.

KEY IDEA:

To solve

$$ay'' + by' + cy = g(t)$$

we need BOTH y_h and y_p. Once we have them, the general solution is the sum:

$$y(t) = y_h(t) + y_p(t)$$

(The homogeneous part and the particular part of the solution).
Secondary Key Idea: $y_p(t)$ can be found by breaking $g(t)$ into a sum of like functions (we'll see examples).
Sections 3.1-3.4: Compute y_h
Sections 3.5, 3.6: How to find y_p.

"The derivative of a polynomial is a polynomial."
Guess the form of $y_p(t)$ for the DE:

$$y'' + 2y' + y = t^2 - 1$$

ANSWER: Guess a general polynomial of degree 2: $y = At^2 + Bt + C$
SUBSTITUTE into the DE to solve for A, B, C:

$$(2A) + 2(2At + B) + (At^2 + Bt + C) = t^2 - 1$$

Now, equate coefficients on both sides:

$$\begin{array}{l|llll} \text{Coeff for } t^2 & A & & & = 1 \\ \text{Coeff for } t & 4A & +B & & = 0 \\ \text{Constants} & 2A & +2B & +C & = -1 \end{array} \qquad A = 1, B = -4, C = 5$$

The particular part of the solution is: $y_p(t) = t^2 - 4t + 5$.
What is the general solution?$\mathrm{e}^{-t}(C_1 + C_2 t) + t^2 - 4t + 5$

This is called the "Method of Undetermined Coefficients":

Guess the *form* of the particular solution. Substitute into the DE solving for the constants.

We have seen that:
If $g(t)$ is a polynomial of degree n, we guess that y_p is a polynomial of degree n.

Suppose $g(t) = \mathrm{e}^{3t}\cos(2t) + t\sin(2t)$. What would we do for our guess?

Example

SOLUTION: First break up the guess as $y_{p_1}(t)$ and $y_{p_2}(t)$.Then

$$y_{p_1} = \mathrm{e}^{3t}(A\cos(2t) + B\sin(2t))$$

and for the second, guess a full linear polynomial times a sine AND cosine:

$$y_{p_2} = (At + B)\sin(2t) + (Ct + D)\cos(2t)$$

How would your guess change if $g(t) = \mathrm{e}^{3t}\cos(2t) + t\mathrm{e}^{3t}\sin(2t)$?
Now we would NOT split the guess:

$$y_p = \mathrm{e}^{3t}((At + B)\cos(2t) + (Ct + D)\sin(2t))$$

Suppose $g(t) = t^2 e^{3t}$. What would we do for our guess?

Example

SOLUTION:

$$y_p = e^{3t}(At^2 + Bt + C)$$

The Method of Undetermined Coefficients

To find the particular solution, we will guess that its form is the same as $g(t)$ (Also see table in text):

$g_i(t)$ is :	The ansatz for y_{p_i} :
$P_n(t)$	$a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0$
$P_n(t)\mathrm{e}^{\alpha t}$	$\mathrm{e}^{\alpha t}(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)\mathrm{e}^{\alpha t} \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$	$\mathrm{e}^{\alpha t}\cos(\beta t)(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0) + \mathrm{e}^{\alpha t}\sin(\beta t)(b_n t^n + \ldots + b_2 t^2 + b_1 t + b_0)$

$$\text{Solve } y'' + 2y' + y = \mathrm{e}^{-t}$$

Problem: With the ansatz $y_p = A\mathrm{e}^{-t}$, we get

$$0 = \mathrm{e}^{-t}$$

The solution: Multiply the ansatz by t until it is no longer part of the homogeneous solution (e.g., until $L(y_p) \neq 0$).
Since $y_h(t) = C_1\mathrm{e}^{-t} + C_2 t\mathrm{e}^{-t}$, we will need to multiply by t^2. Our ansatz is now

$$y_p = At^2\mathrm{e}^{-t}$$

Note: Not a full second degree polynomial. Substitution yields $A = 1/2$, so the solution is:

$$y(t) = \mathrm{e}^{-t}\left(C_1 + C_2 t + \frac{1}{2}t^2\right)$$

Example

Let $y'' - y' - 2y = -4t\mathrm{e}^t + \mathrm{e}^{2t}$. Give your (final) ansatz: First, get y_h for comparison later:

$$r = -1, 2 \quad \Rightarrow \quad y_h(t) = C_1\mathrm{e}^{-t} + C_2\mathrm{e}^{2t}$$

Now, for the first part of g, guess a degree 1 poly times exponential:

$$y_{p_1}(t) = (At + B)\mathrm{e}^t$$

Check against the homogeneous solution: $t\mathrm{e}^t$ and e^t are NOT solutions. Now for the second function:

$$y_{p_2}(t) = C\mathrm{e}^{2t}$$

We see e^{2t} does solve $L(y) = 0$, so multiply by t: $y_{p_2}(t) = Ct\mathrm{e}^{2t}$
The final ansatz for the form of the particular solution is then:

$$(At + B)\mathrm{e}^t + Ct\mathrm{e}^{2t}$$

The final version of the Method (Same as the Table in the text):

The Method of Undetermined Coefficients

To find the particular solution, we will guess that its form is the same as $g(t)$ (Also see table in text):

$g_i(t)$ is :	The ansatz for y_{p_i} :
$P_n(t)$	$t^s(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)\mathrm{e}^{\alpha t}$	$t^s \mathrm{e}^{\alpha t}(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0)$
$P_n(t)\mathrm{e}^{\alpha t} \begin{cases} \sin(\beta t) \\ \cos(\beta t) \end{cases}$	$t^s \mathrm{e}^{\alpha t} \cos(\beta t)(a_n t^n + \ldots + a_2 t^2 + a_1 t + a_0) +$ $t^s \mathrm{e}^{\alpha t} \sin(\beta t)(b_n t^n + \ldots + b_2 t^2 + b_1 t + b_0)$

where $s = 0, 1,$ or 2.

Example

Give the ansatz for the particular part of the solution, if

$$y'' + 2y' + y = t\mathrm{e}^t \sin(2t)$$

First, check y_h:

$$r = -1, -1 \quad \Rightarrow \quad y_h = \mathrm{e}^{-t}(C_1 + C_2 t)$$

For y_p: Is this correct?

$$y_p(t) = \mathrm{e}^t (At + B)(C\sin(t) + D\cos(t))$$

No! The table says that we need a polynomial for each sine and cosine. That is,

$$y_p(t) = \mathrm{e}^t \left((At + B)\sin(2t) + (Ct + D)\cos(2t)\right)$$

For each DE with the given forcing function, give the (final) form of the ansatz. For your convenience, the roots to the characteristic equation are also provided:

- $g(t) = 3t$ with $r = 0, -1$ SOLN: $y_p = t(At + B)$
- $g(t) = t\sin(3t) + \mathrm{e}^{2t}$ with $r = 2, 3$
  $y_p = (At + B)\sin(3t) + (Ct + D)\cos(3t) + At\mathrm{e}^{2t}$
- $g(t) = 3\cos(2t)$ with $r = -1 \pm 2i$

$$y_p = A\cos(2t) + B\sin(2t)$$

- $g(t) = \cos(\omega t)$ with $r = \pm\omega i$

$$y_p = t(A\sin(\omega t) + B\cos(\omega t))$$