Solutions (Last HW Set)

1. For each differential equation below, (i) convert into an equivalent system of first order differential equations, (ii) Show that the characteristic equation we get from Chapter 3 is the same as the one for the eigenvalues. (iii) Classify the origin's stability (using Poincare), and (iv) Solve the equation (and the system).

   (a) $y'' - y' - 6y = 0$

   SOLUTION: Let $x_1 = y$ and $x_2 = y'$. Then the system of DEs becomes:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= 6x_1 + x_2 \end{aligned} \quad \Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 - \lambda - 6 = 0 \qquad \lambda = -2, 3$$

   This is the same as the usual characteristic equation, $r^2 - r - 6 = 0$. From the eigenvalues, we can tell that the origin is a saddle (we could also use the Poincare diagram). Solving the original equation:

   $$y(t) = C_1 \mathrm{e}^{-2t} + C_2 \mathrm{e}^{3t}$$

   Notice that to solve the system, $x_1 = y$ and $x_2 = y'$, so that (in vector form), the solution to the system is:

   $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \mathrm{e}^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + C_2 \mathrm{e}^{3t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

   (b) $y'' - 2y' + y = 0$

   Solving this the same way as the first one: Let $x_1 = y$ and $x_2 = y'$. Then the system of DEs becomes:

   $$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + 2x_2 \end{aligned} \quad \Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \qquad \lambda = 1, 1$$

   This is the same as the usual characteristic equation, $r^2 - 2r + 1 = 0$. From the eigenvalues, we can tell that the origin is a degenerate source (we could also use the Poincare diagram). Solving the original equation:

   $$y(t) = C_1 \mathrm{e}^{t} + C_2 t \mathrm{e}^{t}$$

   Notice that to solve the system, $x_1 = y$ and $x_2 = y'$, so that (in vector form), the solution to the system is:

   $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = C_1 \mathrm{e}^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 \mathrm{e}^{t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

   From here you can see that $[1, 1]^T$ is the eigenvector and $[0, 1]^T$ is a generalized eigenvector for the matrix.

(c) $y'' + 2y' + 5y = 0$

SOLUTION: Continue as before-Let $x_1 = y$ and $x_2 = y'$. Then the system of DEs becomes:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -5x_1 - 2x_2 \end{aligned} \quad \Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} \quad \Rightarrow \quad \lambda^2+2\lambda+5=0 \qquad \lambda = -1+2i$$

This is the same as the usual characteristic equation, $r^2 + 2r + 5 = 0$. From the eigenvalues, we can tell that the origin is a spiral sink (we could also use the Poincare diagram). Solving the original equation:

$$y(t) = C_1 \mathrm{e}^{-t}\cos(2t) + C_2\mathrm{e}^{-t}\sin(2t)$$

Notice that to solve the system, $x_1 = y$ and $x_2 = y'$, so that (in vector form), the solution to the system is:

$$\begin{aligned} x_1 &= C_1\mathrm{e}^{-t}\cos(2t) + C_2\mathrm{e}^{-t}\sin(2t) \\ x_2 &= -C_1\mathrm{e}^{-t}\cos(2t) - 2C_1\mathrm{e}^{-t}\sin(2t) - C_2\mathrm{e}^{-t}\sin(2t) + 2C_2\cos(2t) \end{aligned}$$

Simplifying:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathrm{e}^{-t}\left(C_1 \begin{bmatrix} \cos(2t) \\ -\cos(2t) - 2\sin(2t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) \\ -\sin(2t) + 2\cos(2t) \end{bmatrix} \right)$$

Just for fun, you should see if you get the same thing by first computing the eigenvector, etc. Here are the first few steps:

$$(0 - (-1+2i))v_1 + v_2 = 0 \quad \Rightarrow \quad \begin{aligned} v_1 &= v_1 \\ v_2 &= (-1+2i)v_1 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 1 \\ -1+2i \end{bmatrix}$$

Now, if we compute $\mathrm{e}^{\lambda t}\mathbf{v}$, we get:

$$\mathrm{e}^{-1+2i)t} \begin{bmatrix} 1 \\ -1+2i \end{bmatrix} = \mathrm{e}^{-t}(\cos(2t) + i\sin(2t)) \begin{bmatrix} 1 \\ -1+2i \end{bmatrix} =$$

$$\mathrm{e}^{-t} \begin{bmatrix} \cos(2t) + i\sin(2t) \\ -\cos(2t) - 2\sin(2t) - i\sin(2t) + 2i\cos(2t) \end{bmatrix}$$

2. For each matrix A below, solve the system $\mathbf{x}' = A\mathbf{x}$ by first converting the system into a second order linear homogeneous DE with constant coefficients (then solve that).

(a) $\begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x_1' &= 2x_1 - x_2 \\ x_2' &= 3x_1 - 2x_2 \end{aligned}$

SOLUTION: From the first equation, $x_2 = -x_1' + 2x_1$. Substituting it into the second equation, we get

$$-x_1''+2x_1' = 3x_1-2(-x_1'+2x_1) \quad \Rightarrow \quad x_1''-x_1 = 0 \quad \Rightarrow \quad r^2-1=0 \quad \Rightarrow \quad r = \pm 1$$

Therefore,

$$x_1 = C_1\mathrm{e}^{t} + C_2\mathrm{e}^{-t} \quad \text{and} \quad x_2 = -x_1' + 2x_1 = C_1\mathrm{e}^{t} + 3C_2\mathrm{e}^{-t}$$

For fun, we could write this in vector form as well:

$$\mathbf{x}(t) = C_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

It's easy to verify that $[1,1]^T$ is an eigenvector for $\lambda = 1$ and $[1,3]^T$ is an eigenvector for $\lambda = -1$

(b) $\begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x_1' & = x_1 + 2x_2 \\ x_2' & = -5x_1 - x_2 \end{array} \quad \Rightarrow \quad x_2 = \frac{1}{2}(x_1' - x_1)$

Substituting into the second equation,

$$x_1'' + 9x_1 = 0 \quad \Rightarrow \quad r^2 + 9 = 0 \quad \Rightarrow \quad r = \pm 3i$$

Therefore,

$$x_1 = C_1 \cos(3t) + C_2 \sin(3t)$$

And, since $x_2 = \frac{1}{2}(x_1' - x_1)$, we have:

$$x_2 = \frac{1}{2}\left(C_1(-\cos(3t) - 3\sin(3t) + C_2(-\sin(3t) + 3\cos(3t))\right)$$

For extra fun, we might express this in vector form and compare to the eigenvalues/eigenvector solution to the system:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} \cos(3t) \\ -(1/2)\cos(3t) - (3/2)\sin(3t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(3t) \\ -(1/2)\sin(3t) + (3/2)\cos(3t) \end{bmatrix}$$

3. For each matrix A below, give the solution to $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors.

(a) $\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

SOLUTION: $\mathbf{x}(t) = e^{2t}\left(C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2\left(t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right)$

(b) $\begin{bmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{bmatrix} \quad \Rightarrow \quad \lambda = -\frac{1}{2} + i \quad \mathbf{v} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\mathbf{x}(t) = e^{-t/2}\left(C_1 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} + C_2 \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}\right)$

(c) $\begin{bmatrix} -1 & -1 \\ 0 & -\frac{1}{4} \end{bmatrix}$

$\mathbf{x}(t) = C_1 e^{-t/4} \begin{bmatrix} -4 \\ 3 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \quad \Rightarrow \quad \lambda = 1 + 2i \quad \mathbf{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$\mathbf{x}(t) = e^t\left(C_1 \begin{bmatrix} \cos(2t) - \sin(2t) \\ 2\cos(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(2t) + \cos(2t) \\ 2\sin(t) \end{bmatrix}\right)$

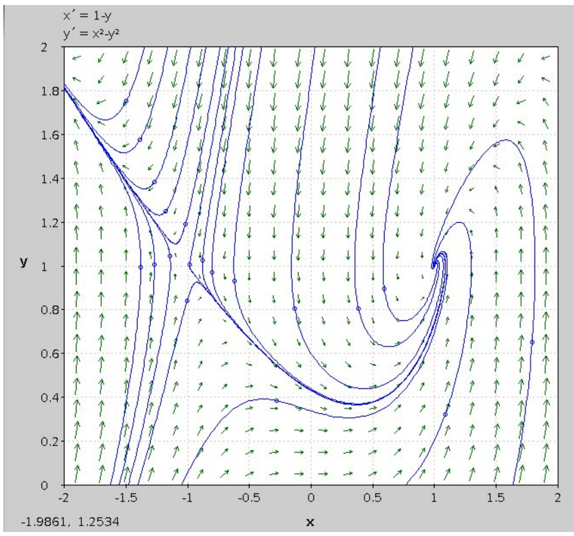
4. For each system below, (i) Find all equilibrium solutions, (ii) Linearize the DE about each, and (iii) Use the Poincaré Diagram to analyze the stability. Finally, go online to the phase plane plotter and see if your analysis is correct.

(a) $\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$

SOLUTION: The equilibria are $(1, 1)$ and $(-1, 1)$. The Jacobian matrix is:

$$J = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix} \qquad J(1,1) = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix} \qquad J(1,-1) = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$$

Therefore, at $(1, 1)$ we have a spiral sink and at $(1, -1)$ there is a saddle.



(b) $\begin{aligned} x' &= \cos(y) \\ y' &= \sin(x) \end{aligned}$

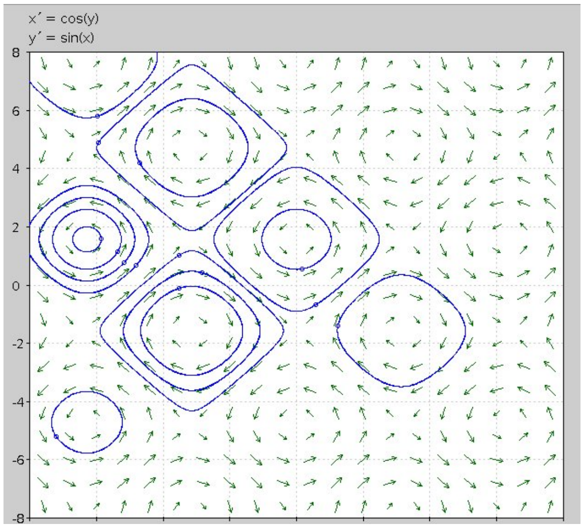
The $\cos(y)$ is zero at odd multiples of $\pi/2$, and $\sin(x)$ is zero at integer multiples of π. Looking at the Jacobian,

$$J = \begin{bmatrix} 0 & -\sin(y) \\ \cos(x) & 0 \end{bmatrix}$$

There are four possible matrices:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

For example, the first matrix is where $\sin(y) = -1$ (if $y = 3\pi/2, 7\pi/2, 11\pi/2$, etc) and $\cos(x) = 1$ (at $x = 0, 2\pi, 4\pi$, etc). The classification for each point then: The first and last matrices correspond to saddles, and the middle two matrices correspond to centers.



(c) $$\begin{aligned} x' &= (2+x)(y-x) \\ y' &= (4-x)(y+x) \end{aligned}$$

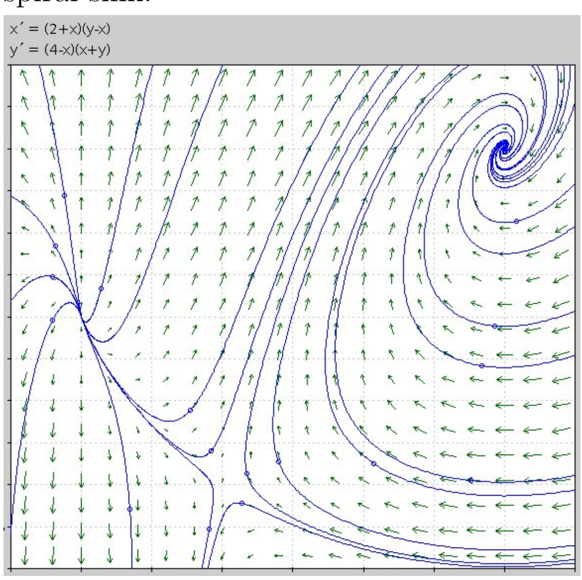
The equilibria are $(0,0)$, $(-2,2)$ and $(4,4)$. The Jacobian matrix is then:

$$J = \begin{bmatrix} y-2x-2 & 2+x \\ -y-2x+4 & 4-x \end{bmatrix}$$

Evaluated at the equilibria:

$$J(0,0) = \begin{bmatrix} -2 & 2 \\ 4 & 4 \end{bmatrix} \qquad J(-2,2) = \begin{bmatrix} 4 & 0 \\ 6 & 6 \end{bmatrix} \qquad J(4,4) = \begin{bmatrix} -6 & 6 \\ -8 & 0 \end{bmatrix}$$

Using a Poincare diagram, these points are (in order): A saddle, a source, and a spiral sink.



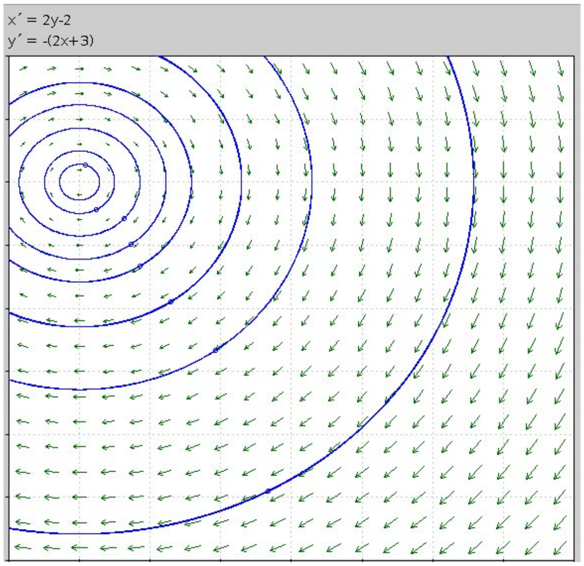
5. Sometimes nonlinear differential equations can be solved by means of methods from Chapter 2, where we convert the system using $dx/dt, dy/dt$ to dy/dx (we've already done a few of these). Here are some more (one linear one just for fun), as a start to your review for the final! Remember, each DE may be classified in multiple ways, so if you get stuck using one technique, you might try another.

(a) $\begin{aligned} x' &= 2y - 2 \\ y' &= -(2x+3) \end{aligned} \quad \Rightarrow \quad (2y-2)\,dy = -(2x+3)\,dx$

Therefore, this is separable (and exact), and the implicit solution is:

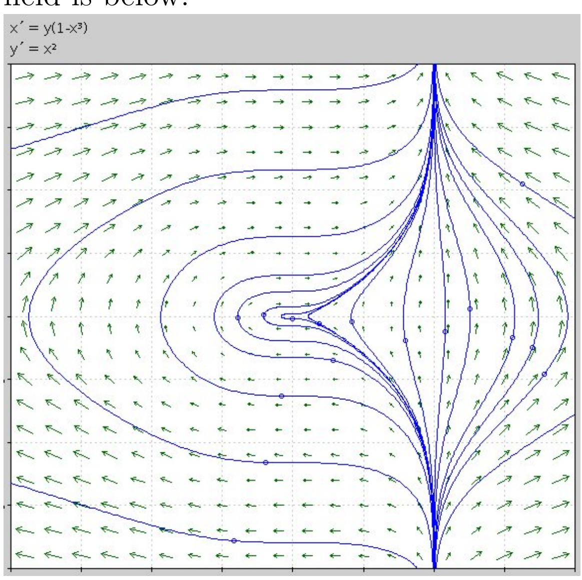
$$x^2 + 3x + y^2 - 2y = C$$

I've written the solution to emphasize that these solutions are circles. Here is the direction field:



(b) $\begin{aligned} x' &= y(1-x^3) \\ y' &= x^2 \end{aligned} \quad \Rightarrow \quad y\,dy = \dfrac{x^2}{1-x^3}\,dx$

The overall implicit solution is therefore $\frac{1}{2}y^2 = -\frac{1}{3}\ln|1-x^3| + C$, and the direction field is below.



(c) $\begin{aligned} x' &= x - y \\ y' &= y - 4x \end{aligned}$

This is homogeneous:

$$\frac{dy}{dx} = \frac{\frac{y}{x} - 4}{1 - \frac{y}{x}}$$

So we substitute $v = \frac{y}{x}$ or $y = xv$, and $y' = xv' + v$:

$$xv' + v = \frac{v-4}{1-v} \quad \Rightarrow \quad \frac{1-v}{v^2-4}\,dv = \frac{1}{x}\,dx$$

Now we can use partial fractions on the expression to the left:

$$\frac{1-v}{v^2-4} = \frac{A}{v-2} + \frac{B}{v+2} = -\frac{1}{4}\frac{1}{v-2} - \frac{3}{4}\frac{1}{v+2}$$

so that:

$$-\frac{1}{4}\ln|v-2| - \frac{3}{4}\ln|v+2| = \ln|x| + C$$

Remember to back-substitute; simplification doesn't yield much info:

$$-\frac{1}{4}\ln|(y/x)-2| - \frac{3}{4}\ln|(y/x)+2| = \ln|x| + C$$

(d) $\begin{aligned} x' &= \mathrm{e}^x\cos(y) + 2\cos(x) \\ y' &= 2y\sin(x) - \mathrm{e}^x\sin(y) \end{aligned}$

This one is exact:

$$-(2y\sin(x) - \mathrm{e}^x\sin(y))\,dx + (\mathrm{e}^x\cos(y) + 2\cos(x))\,dy = 0$$

so that $M_y = -2\sin(x) + \mathrm{e}^x\cos(y) = N_x$. Now, using M for the antiderivative,

$$f(x,y) = 2y\cos(x) + \mathrm{e}^x\sin(y) + H(y)$$

and differentiating with respect to y, we get N:

$$f_y = 2\cos(x) + \mathrm{e}^x\cos(y) + H'(y)$$

so $H'(y) = 0$, and we can now form the solution:

$$2y\cos(x) + \mathrm{e}^x\sin(y) = C$$