M244: Solutions to Final Exam Review

1. Solve (use any method if not otherwise specified):

   (a) $(2x-3x^2)\frac{dx}{dt} = t\cos(t)$
   SOLUTION: As a separable DE:
   $$\int 2x - 3x^2\,dx = \int t\cos(t)\,dt \Rightarrow x^2 - x^3 = \cos(t) + t\sin(t) + C$$
   You can leave your answer in implicit form.

   (b) $y'' + 2y' + y = \sin(3t)$
   SOLUTION: Get the homogeneous part then the particular solution (or use Laplace):
   $$r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1, -1 \Rightarrow y_h(t) = \mathrm{e}^{-t}\left(C_1 + C_2 t\right)$$
   For the particular solution, (Undet Coefs), $y_p = A\cos(3t) + B\sin(3t)$. Substitute to get:
   $$(A - 6B - 9A)\sin(3t) + (B + 6A - 9B)\cos(3t) = \sin(3t)$$
   so that $A = -3/50$, $B = -2/25$. Altogether,
   $$y(t) = \mathrm{e}^{-t}\left(C_1 + C_2 t\right) - \frac{3}{50}\cos(3t) - \frac{2}{25}\sin(3t)$$

   (c) $y'' - 3y' + 2y = \mathrm{e}^{2t}$
   SOLUTION: Same technique here. The roots to the characteristic equation are $r = 1, 2$, so the homogeneous part of the solution is:
   $$y_h(t) = C_1\mathrm{e}^t + C_2\mathrm{e}^{2t}$$
   Initially, we guess that $y_p(t) = A\mathrm{e}^{2t}$, but that is part of y_h, so multiply by t: $y_p = At\mathrm{e}^{2t}$. Now substitute into the D.E. to get: $A = 1$. The full solution is
   $$y(t) = C_1\mathrm{e}^t + C_2\mathrm{e}^{2t} + t\mathrm{e}^{2t}$$

   (d) $y' = \sqrt{t}\mathrm{e}^{-t} - y$.
   SOLUTION: This is a linear differential equation, with integrating factor; $y' + y = \sqrt{t}\mathrm{e}^{-t}$. The integrating factor is $\mathrm{e}^{\int 1\,dt} = \mathrm{e}^t$. Therefore,
   $$(y\mathrm{e}^t)' = \sqrt{t} \Rightarrow y\mathrm{e}^t = \frac{2}{3}t^{3/2} + C \Rightarrow y = \left(\frac{2}{3}t^{3/2} + C\right)\mathrm{e}^{-t}$$

   (e) $x' = 2 + 2t^2 + x + t^2x$.
   SOLUTION: This is a linear differential equation: $x' - (1+t^2)x = 2(1+t^2)$. The integrating factor is:
   $$\mathrm{e}^{\int -(1+t^2)\,dt} = \mathrm{e}^{-t-(1/3)t^3}$$
   So we solve the following (to integrate, let $u = t + (1/3)t^2$)
   $$\left(x\mathrm{e}^{-t-(1/3)t^3}\right)' = 2(1+t^2)\mathrm{e}^{-t-(1/3)t^3} \Rightarrow \left(x\mathrm{e}^{-t-(1/3)t^3}\right) =$$
   $$-2\mathrm{e}^{-t-(1/3)t^3} + C \Rightarrow x = -2 + C\mathrm{e}^{t+(1/3)t^3}$$

2. Show that with the proper substitution, the following equation becomes separable. NOTE: You do not need to solve the differential equation.
   $$\frac{dy}{dx} = \frac{3x-4y}{y-2x}$$
   SOLUTION: Divide numerator and denominator by x, and let $v = \frac{y}{x}$ (the equation is a homogeneous equation), therefore $y = xv$ and $y' = xv' + v$. Although we don't need to solve the DE, we should go ahead and separate the variables.
   $$xv' + v = \frac{3-4v}{v-2} \quad \Rightarrow \quad xv' = \frac{3-4v-v(v-2)}{v-2} = \frac{3-2v-v^2}{v-2} \quad \Rightarrow \quad \frac{v-2}{3-2v-v^2}\,dv = \frac{1}{x}\,dx$$

3. Show that with the proper substitution, the following equation becomes linear. NOTE: You do not need to solve the differential equation.

$$\frac{dy}{dx} + 3xy = \frac{x}{y^2}$$

SOLUTION: This is a Bernoulli equation. Multiplying through by y^2 gives:

$$y^2 y' + 3xy^3 = x$$

Let $v = y^3$. Then $v' = 3y^2 y'$, or $\frac{1}{3}v' = y^2 y'$. Therefore, we get:

$$\frac{1}{3}v' + 3xv = x \quad \Rightarrow \quad v' + 9xv = 3x$$

which is linear in v.

4. Obtain the general solution in terms of α, then determine a value of α so that $y(t) \to 0$ as $t \to \infty$:

SOLUTION:

$$y'' - y' - 6y = 0, \quad y(0) = 1, y'(0) = \alpha$$

The general solution (before initial conditions):

$$y(t) = C_1 \mathrm{e}^{3t} + C_2 \mathrm{e}^{-2t}$$

With the initial conditions,

$$1 = C_1 + C_2 \quad \alpha = 3C_1 - 2C_2 \Rightarrow C_1 = \frac{2+\alpha}{5}, \quad C_2 = \frac{3-\alpha}{5}$$

Therefore,

$$y(t) = \frac{2+\alpha}{5}\mathrm{e}^{3t} + \frac{3-\alpha}{5}\mathrm{e}^{-2t}$$

For $y(t) \to 0$, we must have $\alpha = -2$ (to zero out the first term).

5. If $y' = y(1-y)(2-y)(3-y)(4-y)$ and $y(0) = 2.5$, determine what y does as $t \to \infty$.

SOLUTION: The idea here is to figure out where the equilibria are and if they are stable or unstable. This is a fifth degree polynomial which increases after $y = 4$ and decreases to $-inf$ as y decreases. From a quick sketch of the polynomial, we see that $y = 1, 3$ are stable and $y = 0, 2, 4$ are unstable equilibria. If the solution starts at $y = 2.5$, it will increase to the next stable equilibrium $y = 3$.

6. If y_1, y_2 are a fundamental set of solutions to

$$t^2 y'' - 2y' + (3+t)y = 0$$

and if $W(y_1, y_2) = 3$, find $W(y_1, y_2)(4)$.

Typo: The line $W(y_1, y_2) = 3$ should read: $W(y_1, y_2)(2) = 3$.

SOLUTION: Before using Abel's Theorem, but the equation in standard form as:

$$y'' - \frac{2}{t^2}y' + \frac{3+t}{t^2}y = 0$$

so that the Wronskian between any two solutions is:

$$C\mathrm{e}^{2\int t^{-2}\,dt} = C\mathrm{e}^{-2/t}$$

Given that the Wronskian is 3 at $t = 2$, we have:

$$C\mathrm{e}^{-1} = 3 \quad \Rightarrow \quad C = 3\mathrm{e}$$

and now

$$W(y_1, y_2)(4) = 3\mathrm{e}\mathrm{e}^{-2/4} = 3\sqrt{e}$$

7. Let $y''' - y' = t\mathrm{e}^{-t} + 2\cos(t)$. First, use our ansatz to find the characteristic equation for the third order homogeneous equation. Determine a suitable form for the particular solution, y_p using Undetermined Coefficients. Do not solve for the coeffs.

   SOLUTION: The ansatz was $y = \mathrm{e}^{rt}$, so that $y' = r\mathrm{e}^{rt}$ and $y''' = r^3\mathrm{e}^{rt}$. Therefore, the homogeneous equation becomes:

   $$y''' - y' = 0 \Rightarrow \mathrm{e}^{rt}\left(r^3 - r\right) = 0$$

   so that $r(r^2 - 1) = 0$. Therefore, $r = 0$, $r = \pm 1$. Extrapolating from the second order differential equation, we expect the homogeneous solution to be:

   $$y_h = C_1 + C_2\mathrm{e}^t + C_3\mathrm{e}^{-t}$$

   and the form for the particular solution (break into two pieces):

   $$y_{p_1} = (At + B)\mathrm{e}^{-t} \Rightarrow y_{p_1} = t(At + B)\mathrm{e}^{-t}$$

   and

   $$y_{p_2} = A\cos(t) + B\sin(t)$$

8. What is the Wronskian? How is it used?

   SOLUTION: The Wronskian is an operation that is performed on two functions f, g:

   $$W(f, g) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix}$$

   The Wronskian is primarily used to determine if two solutions to a second order linear homogeneous differential equation form a *fundamental set* of solutions- That is, if an arbitrary solution to an IVP could be written as a linear combination of the functions in the set.

9. Explain Abel's Theorem:

   SOLUTION: Abel's Theorem states that, given any two solutions to a second order linear homogeneous DE (on the interval on which the existence and uniqueness theorem holds):

   $$y'' + p(t)y' + q(t)y = 0$$

   Then the Wronskian can be computed by:

   $$W(y_1, y_2)(t) = C\mathrm{e}^{-\int p(t)\,dt}$$

   This implies that either the Wronskian is always zero on the interval, or never zero (the two functions form a fundamental set).

10. Give the three Existence and Uniqueness Theorems we have had in class.

    SOLUTION: We had the general existence and uniqueness theorem, then one for linear first order, then one for linear second order equations:

    - Let $y' = f(t, y)$ with $y(t_0) = y_0$. If there is a open rectangle R that contains (t_0, y_0), and on which f and $\partial f/\partial y$ are continuous, then there exists an $\epsilon > 0$ for which a unique solution exists to the DE, valid for $t_0 - \epsilon < t < t_0 + \epsilon$.
    - Let $y' + p(t)y = g(t)$, with $y(t_0) = y_0$. If there is an open interval I containing t_0 on which both p, g are continuous, then there is a solution to the IVP, valid on all of I.
    - Let $y'' + p(t)y' + q(t)y = g(t)$, with $y(t_0) = y_0, y'(t_0) = v_0$. If there is an open interval I containing t_0 on which both p, q, g are continuous, then there is a solution to the IVP, valid on all of I.

11. Let $y'' - 6y' + 9y = F(t)$. For each $F(t)$ listed, give the *form* of the general solution using undet. coeffs (do not solve for the coefficients).

    SOLUTION: Before we start, we should go ahead and solve the homogeneous equation:

    $$r^2 - 3r + 9 = 0 \quad \Rightarrow \quad (r - 3)^2 = 0 \quad \Rightarrow \quad r = 3, 3$$

    Therefore,

    $$y_h = C_1\mathrm{e}^{3t} + C_2t\mathrm{e}^{3t}$$

(a) $F(t) = 2t^2$ SOLUTION: Guess the full quadratic, $y_p = At^2 + Bt + C$

(b) $F(t) = t\mathrm{e}^{-3t}\sin(2t)$: SOLUTION: We need a polynomial of degree 1 with each sine and cosine:

$$y_p(t) = \mathrm{e}^{-3t}((At + B)\cos(2t) + (Ct + D)\sin(2t))$$

(c) $F(t) = t\sin(2t) + \cos(2t)$ SOLUTION: Similar to the last one, but we can take both sine and cosine together:

$$y_p = (At + B)\cos(2t) + (Ct + D)\sin(2t)$$

(d) $F(t) = 2t^2 + 12\mathrm{e}^{3t}$ SOLUTION: Break this one up into two solutions- One for $g_1 = t^2$ and one for $g_2 = 12\mathrm{e}^{3t}$. Then:

$$y_{p_1}(t) = At^2 + Bt + C \qquad y_{p_2} = At^2\mathrm{e}^{3t}$$

where I multiplied the second guess by t^2 so that it looks like no term of the homogeneous part of the solution.

12. Consider the predator-prey system:

$$\begin{aligned} x' &= x(1 - 0.5x - 0.5y) \\ y' &= y(-0.25 + 0.5x) \end{aligned}$$

Typo: The second differential equation should read: $y(-0.25 + 0.5\mathbf{x})$.

(a) Which is predator, which is prey? How are the populations being modeled (what assumptions)?
SOLUTION: Since the population x decreases and population y increases with interactions, x is the prey and y is the predator. In modeling the prey, we assume that, in absence of the prey, there is a logistic equation governing growth. Similarly, with no prey, we assume that the rate of change of the predator population decreases at a rate proportional to its current size. In introducing the other, the rate of change of the prey decreases at a rate proportional to the number of predator-prey interactions, and the rate of change of the population of the predator increases (at a rate proportional to the number of interactions).

(b) Find the equilibrium solutions.
SOLUTION:

$$(0, 0) \quad (2, 0) \quad \left(\frac{1}{2}, \frac{3}{2}\right)$$

(c) Classify the equilibria using the Poincaré diagram.
SOLUTION: The Jacobian matrix is:

$$\begin{bmatrix} 1 - x - 0.5y & -0.5x \\ 0.5y & -0.25 + 0.5x \end{bmatrix}$$

In order, linearizing at each equilibrium yields the following matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & -0.25 \end{bmatrix} \qquad \begin{bmatrix} -1 & -1 \\ 0 & 0.75 \end{bmatrix} \qquad \begin{bmatrix} -0.25 & -0.25 \\ 0.75 & 0 \end{bmatrix}$$

The first and second points correspond to SADDLEs, and the third point is a SPIRAL SINK.

(d) What will happen to the population $x(t)$ for almost every starting point (as $t \to \infty$)?
Almost every starting point for the two populations will oscillate around and get closer to the (stable) equilibrium at $(1/2, 3/2)$. This is a very stable model.

13. Consider the system:

$$\begin{aligned} x' &= \cos(y) \\ y' &= \sin(x) \end{aligned}$$

See if you can sketch the direction field (find/classify the equilibria first!).

SOLUTION: See the homework solutions to Chapter 9.

14. A spring is stretched 0.1 m by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 0.05 m below its resting equilibrium and released with a downward velocity of 0.1 m/s, determine its position u at time t.

    SOLUTION: From what is given, we can compute the spring constant.

$$mg - kL = 0 \quad \Rightarrow \quad 3 - \frac{k}{10} = 0 \quad \Rightarrow \quad k = 30$$

    Now, we also assume that damping is proportional to the velocity, so:

$$\gamma v = 5\gamma = 3 \quad \Rightarrow \quad \gamma = \frac{3}{5}$$

    Now we can construct the model (downward is positive):

$$2u'' + \frac{3}{5}u + 30u = 0 \qquad u(0) = 0.05 \qquad u'(0) = 0.1$$

    We'll go ahead and use some numerical approximations for this- The roots of the characteristic equation are approximately:

$$r = -0.15 \pm 3.87\,i$$

    (Sorry! This came out of Section 3.7 so I thought the numbers would work out better...) Therefore, the solution is:

$$\mathrm{e}^{-0.15t}(C_1\cos(3.87\,t) + C_2\sin(3.87\,t))$$

    If we put in the initial conditions, we get:

$$C_1 = 0.05 \qquad -0.15C_1 + 3.87C_2 = 0.1 \quad \Rightarrow \quad C_2 \approx 0.065$$

    which gives the solution.

15. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$. Find the recurrence relation and write the first 5 terms of the expansion of y.

    Same idea as the previous exercise, but be careful to use powers of $(x-1)$!

$$\sum_{n=2}^{\infty} n(n-1)c_{n-2}(x-1)^{n-2} - x\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} - \sum_{n=0}^{\infty} c_n(x-1)^n = 0$$

    For the middle sum, recall our trick: $x = 1 + (x-1)$, which gives us a way of incorporating the x into the sum:

$$x\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} = (1+(x-1))\sum_{n=1}^{\infty} nc_n(x-1)^{n-1} = \sum_{n=1}^{\infty} nc_n(x-1)^{n-1} + \sum_{n=1}^{\infty} nc_n(x-1)^n$$

    Now shift the index of every sum to match, and you should get the recurrence relation:

$$C_{k+2} = \frac{1}{k+2}\,(C_{k+1} + C_k)$$

    We get:

$$y(x) = C_0 + C_1(x-1) + \frac{C_0 + C_1}{2}(x-1)^2 + \frac{C_0 + 3C_1}{6}(x-1)^3 + \frac{2C_0 + 3C_1}{12}(x-1)^4 + \dots$$

    We can check our answer using the next exercise...

16. Let $y(x)$ be a power series solution to $y'' - xy' - y = 0,\ x_0 = 1$ (the same as the previous DE), with $y(1) = 1$ and $y'(1) = 2$. Compute the first 5 terms of the power series solution by first computing $y''(1), y'''(1), y^{(4)}(1)$.

First, let's compute the derivatives:

$$y'' = xy' + y \quad \text{at } x = 1 \quad \Rightarrow y''(1) = y'(1) + y(1)$$

so that:

$$y''' = xy'' + 2y' \quad \text{at } x = 1 \quad \Rightarrow y'''(1) = y''(1) + 2y'(1)$$

and:

$$y^{(4)} = xy''' + 3y'' \quad \text{at } x = 1 \quad \Rightarrow y^{(4)}(1) = y'''(1) + 3y''(1)$$

From this, we see that:

$$y''(1) = 3, \qquad y'''(1) = 7, \qquad y^{(4)}(1) = 16$$

Writing out the solution:

$$y(x) = 1 + 2(x-1) + \frac{3}{2}(x-1)^2 + \frac{7}{6}(x-1)^3 + \frac{16}{24}(x-1)^4 + \ldots$$

17. Use the definition of the Laplace transform to determine $\mathcal{L}(f)$:

$$f(t) = \begin{cases} 3, & 0 \le t \le 2 \\ 6-t, & 2 < t \end{cases}$$

SOLUTION:

$$\int_0^\infty f(t)\mathrm{e}^{-st}\,dt = 3\int_0^2 \mathrm{e}^{-st}\,dt + \int_2^\infty (6-t)\mathrm{e}^{-st}\,dt =$$

$$\frac{3}{s}(1-\mathrm{e}^{-2s}) + \frac{\mathrm{e}^{-2s}}{s^2}(4s-1)$$

18. Determine the Laplace transform:

(a) $t^2\mathrm{e}^{-9t} \Rightarrow \dfrac{2}{(s+9)^3}$

(b) $u_5(t)(t-2)^2 \Rightarrow \quad \mathrm{e}^{-5s}\left(\frac{2}{s^2} + \frac{6}{s^2} + \frac{9}{s}\right)$

(c) $\mathrm{e}^{3t}\sin(4t) \Rightarrow \dfrac{4}{(s-3)^2+16}$

(d) $\mathrm{e}^{t}\delta(t-3) \Rightarrow \mathrm{e}^{-3s+3}$

19. Find the inverse Laplace transform:

(a) $\dfrac{2s-1}{s^2-4s+6}$. Rewrite: $2\cdot\dfrac{s-2}{(s-2)^2+2} + \dfrac{3}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{(s-2)^2+2}$ The inverse is then $\mathrm{e}^{2t}\left(2\cos(\sqrt{2}t + \dfrac{3}{\sqrt{2}}\sin(\sqrt{2}t)\right)$

(b) $\dfrac{7}{(s+3)^3} \Rightarrow \dfrac{7}{2}t^2\mathrm{e}^{-3t}$

(c) $\dfrac{\mathrm{e}^{-2s}(4s+2)}{(s-1)(s+2)}$. You might rewrite this as $\mathrm{e}^{-2s}H(s)$, where

$$H(s) = \frac{4s+2}{(s-1)(s+2)} = \frac{2}{s+2} + \frac{2}{s-1}$$

Now, $h(t) = 2\mathrm{e}^{-2t} + 2\mathrm{e}^{t}$, and the solution is $u_2(t)h(t-2)$.

(d) $\dfrac{3s-2}{(s-4)^2-3}$ We might rewrite this as:

$$3\cdot\frac{s-4}{(s-4)^2-3} + \frac{10}{\sqrt{3}}\cdot\frac{\sqrt{3}}{(s-4)^2-3} = 3F(s-4) + \frac{10}{\sqrt{3}}G(s-4)$$

where $F(s) = \dfrac{s}{s^2-3}, G(s) = \dfrac{\sqrt{3}}{s^2-3}$. The inverse is (Item 14 from the Table):

$$\mathrm{e}^{4t}\left(3f(t) + \frac{10}{\sqrt{3}}g(t)\right) = \mathrm{e}^{4t}\left(3\cosh(\sqrt{3}t) + \frac{10}{\sqrt{3}}\sinh(\sqrt{3}t)\right)$$

20. Solve the given initial value problems using Laplace transforms:

(a) $y'' + 2y' + 2y = 4t$, $y(0) = 0$, $y'(0) = -1$. The Laplace transform:

$$Y(s) = \frac{4 - s^2}{s^2(s^2 + 2s + 2)} = -\frac{2}{s} + \frac{2}{s^2} + \frac{2s+1}{s^2 + 2s + 2} =$$

$$-\frac{2}{s} + \frac{2}{s^2} + 2\frac{s+1}{(s+1)^2 + 1} - \frac{1}{(s+1)^2 + 1}$$

so that

$$y(t) = -2 + 2t + \mathrm{e}^{-t}\left(2\cos(t) - \sin(t)\right)$$

(b) $y'' - 2y' - 3y = u_1(t)$, $y(0) = 0$, $y'(0) = -1$ Use partial fractions:

$$Y(s) = -\frac{1}{4} \cdot \frac{1}{s+1} + \frac{1}{4} \cdot \frac{1}{s-3} + \mathrm{e}^{-s}\left(-\frac{1}{3} \cdot \frac{1}{s} + \frac{1}{4} \cdot \frac{1}{s+1} + \frac{1}{12} \cdot \frac{1}{s-3}\right)$$

Think of this second term as $\mathrm{e}^{-s} \cdot H(S)$, where

$$h(t) = -\frac{1}{3} + \frac{1}{4}\mathrm{e}^{-t} + \frac{1}{12}\mathrm{e}^{3t}$$

and the solution is:

$$y(t) = -\frac{1}{4}\mathrm{e}^{-t} + \frac{1}{4}\mathrm{e}^{3t} + u_1(t)h(t-1)$$

(c) $y'' - 4y' + 4y = t^2\mathrm{e}^t$, $y(0) = 0$, $y'(0) = 0$.

$$Y(s) = \frac{2}{(s-1)^3(s-2)^2} = \frac{2}{(s-1)^3} \cdot \frac{1}{(s-2)^2} = F(s)G(S)$$

Where $f(t) = t^2\mathrm{e}^t$ and $g(t) = t\mathrm{e}^{2t}$. Therefore,

$$y(t) = t^2\mathrm{e}^t * t\mathrm{e}^{2t}$$

21. Consider

$$t^2y'' - 4ty' + 6y = 0$$

(a) Thinking of this as an Euler equation, find the solution.
SOLUTION: In the Euler equation, we assumed that $y = t^r$, and substituting this into the DE gave the following, where I divide by t^r to get the characteristic equation:

$$r(r-1)t^r - 4t^r + 6t^r = 0 \quad \Rightarrow \quad r^2 - 5r + 6 = 0 \quad \Rightarrow \quad (r-2)(r-3) = 0$$

so the general solution is

$$C_1t^2 + C_2t^3$$

(b) Using $y_1 = t^2$ as one solution, find y_2 by computing the Wronskian two ways.
SOLUTION: We are verifying the solution above by using the Wronskian. Using this technique, we have:

$$\begin{vmatrix} t^2 & y_2 \\ 2t & y_2' \end{vmatrix} = t^2y_2' - 2ty_2$$

And, using Abel's Theorem (with $p(t) = -\frac{4}{t}$):

$$W = C\mathrm{e}^{\int \frac{4}{t}\,dt} = Ct^4$$

Therefore,

$$t^2y_2' - 2ty_2 = Ct^4 \quad \Rightarrow \quad y_2' - \frac{2}{t}y_2 = Ct^2$$

This has an integrating factor of t^{-2}:

$$\left(\frac{y_2}{t^2}\right)' = C \quad \Rightarrow \quad \frac{y_2}{t^2} = Ct + C_1 \quad \Rightarrow \quad y_2 = Ct^3 + C_1t^2$$

We already have $y_1 = t^2$, so $y_2 = t^3$ (as before).

(c) Using $y_1 = t^2$ as one solution, find y_2 by using Variation of Parameters.

Typo: Not variation of parameters, but reduction of order!

SOLUTION: In the reduction of order, we use the following substitutions:

$$y_2 = vy_1 = t^2 v \quad y_2' = t^2 v + 2tv \quad y_2'' = t^2 v'' + 4tv' + 2v$$

Substituting into the DE gives:

$$\begin{array}{rlll} t^2 y_2'' & = t^4 v'' + & 4t^3 v' + & 2t^2 v \\ -4ty_2' & = - & 4t^3 v' - & 8t^2 v \\ +6y_2 & = & & 6t^2 v \\ \hline 0 & = t^4 v'' & & \end{array} \quad \Rightarrow \quad v'' = 0 \quad \Rightarrow \quad v = C_1 t + C_2$$

from which:

$$y_2 = t^2 v = C_1 t^3 + C_2 t^2$$

and again, since $y_1 = t^2$, we only require that $y_2 = t^3$.

22. For the following differential equations, (i) Give the general solution, (ii) Solve for the specific solution, if its an IVP, (iii) State the interval for which the solution is valid.

(a) $y' - \frac{1}{2}y = e^{2t} \qquad y(0) = 1$. This is a linear (integrating factor) differential equation; the solution will be valid for all time t.

$$y(t) = \frac{2}{3}e^{2t} + \frac{1}{3}e^{\frac{1}{2}t}$$

(b) $y' = \frac{1}{2}y(3 - y)$.
SOLUTION: Separable. We'll need partial fractions to integrate.

$$\int \frac{1}{3} \cdot \frac{1}{y} + \frac{1}{3} \cdot \frac{1}{3-y}\, dy = \frac{1}{2}t + C \Rightarrow \ln|y| - \ln|3-y| = \frac{3}{2}t + C_2$$

Now solve for y:

$$\frac{y}{3-y} = Ae^{3/2t} \Rightarrow y = \frac{3}{1 + Be^{-3/2t}}$$

If the initial condition is positive, this is valid for all time (Draw the phase diagram and direction field for this autonomous DE to see why). If the initial condition is negative, we would need to find where (in positive time) the solution has a vertical asymptote. NOTE: We are assuming that $t \geq 0$.

(c) $y'' + 2y' + y = 0, \quad y(0) = \alpha, y'(0) = 1$

$$y(t) = \alpha e^{-t} + (1 + \alpha)te^{-t}$$

This solution is valid for all t.

(d) $2xy^2 + 2y + (2x^2 y + 2x)y' = 0$ This is an exact equation:

$$\frac{\partial}{\partial y}(2xy^2 + 2y) = 4xy + 2 = \frac{\partial}{\partial x}(2x^2 y + 2x)$$

Recall that the solution will be (implicit) $F(x, y) = C$, where

$$F_x = 2xy^2 + 2y \Rightarrow F(x, y) = x^2 y^2 + 2xy + h(x)$$

and

$$F_y = 2x^2 y + 2x \Rightarrow F(x, y) = x^2 y^2 + 2xy + g(y)$$

Comparing, we see $F(x, y) = x^2 y^2 + 2xy$, and the implicit solution is:

$$x^2 y^2 + 2xy = C$$

Here we will not be able to give an interval on which the solution is valid unless we isolate y, although we would have a requirement that $2x^2 y + 2x \neq 0$, so that y' would be defined.

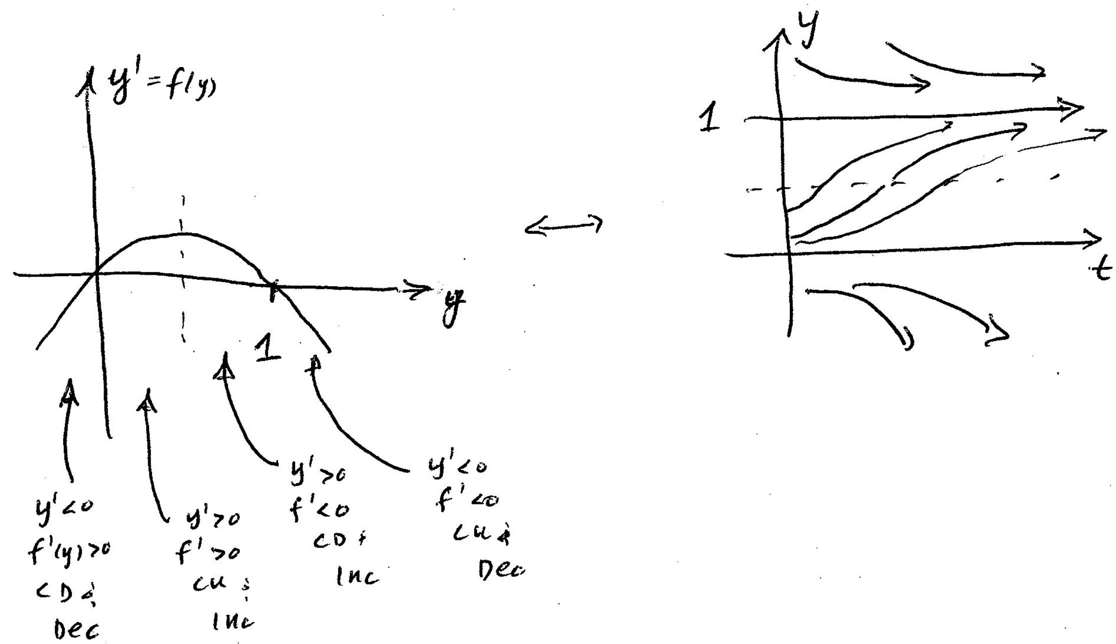
(e) $y'' + 4y = t^2 + 3\mathrm{e}^t, y(0) = 0, y'(0) = 1.$

$$y(t) = \frac{1}{5}\sin(2t) - \frac{19}{40}\cos(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}\mathrm{e}^t$$

The solution is valid for all t.

23. Suppose $y' = -ky(y-1)$, with $k > 0$. Sketch the phase diagram. Find and classify the equilibrium. Draw a sketch of y on the direction field, paying particular attention to where y is increasing/decreasing and concave up/down. Finally, get the analytic (general) solution.

Your graph should be an upside down parabola (vertex up). There are equilibrium solutions at $y = 0$ (unstable) and $y = 1$ (stable).



24. True or False (and explain): Every separable equation is also exact. If true, is one way easier to solve over the other?

True:

$$\frac{dy}{dx} = f(y)g(x) \quad \Rightarrow \quad -g(x)\,dx + \frac{1}{f(y)}dy = 0$$

So, letting $-g(x) = M(x,y)$ and $1/f(y) = N(x,y)$, then $M_y = N_x = 0$. Therefore, every separable equation is exact.

To see if either is simpler, check to see what integrals must be performed. Treating the equation as separable, we have to compute the two integrals:

$$\int \frac{dy}{f(y)} \qquad \int g(x)\,dx$$

Treating the equations as exact,

$$\int M(x,y)\,dx = -\int g(x)\,dx + H(y)$$

$$\int N(x,y)\,dy = \int \frac{dy}{f(y)} + G(x)$$

so we have to compute the same integrals either way (so neither is easier than the other).

25. Let $y' = 2y^2 + xy^2$, $y(0) = 1$. Solve, and find the minimum of y. Hint: Determine the interval for which the solution is valid.

This is separable: $y' = y^2(2 + x) \Rightarrow y^{-2}\,dy = (2 + x)\,dx$, so

$$y(x) = \frac{-2}{x^2 + 4x - 2}$$

This has vertical asymptotes at $x = -2 \pm \sqrt{6}$, so that the solution is valid only when $-2 - \sqrt{6} < x < -2 + \sqrt{6}$, or when x is approximately between -4.45 and 0.45. Between these vertical asymptotes, y has a minimum where its derivative is 0,

$$y' = y^2(2 + x) = 0 \Rightarrow y = 0 \text{ or } x = -2$$

From our solution, we see that $y \neq 0$, so the minimum occurs at $x = -2$, and the minimum is: $-1/5$.

26. A sky diver weighs 180 lbs and falls vertically downward. Assume that the constant for air resistance is 3/4 before the parachute is released, and 12 after it is released at 10 sec. Assume velocity is measured in feet per second, and $g = 32$ ft/sec^2.

(a) Find the velocity of the sky diver at time t (before the parachute opens).
SOLUTION: Recall our model:

$$mv' = mg - \gamma v$$

And, since $180 = mg = 32m$, we get that mass is 45/8. Therefore, the model becomes:

$$v' = 32 - \frac{3}{4} \cdot \frac{8}{45} v = 32 - \frac{2}{15} v \qquad v(0) = 0$$

Then, for the first 10 seconds, the velocity is given by:

$$240 - 240\mathrm{e}^{-(2/15)t}$$

At 10 seconds, the model changes to:

$$v' = 32 - 12 \cdot \frac{8}{45} v = 32 - \frac{32}{15} v$$

The general solution (for $t > 10$) is:

$$C_1 + 15\mathrm{e}^{-(32/15)t}$$

and we would find C_1 so that the velocity function is continuous (a bit of a mess algebraically).

(b) If the sky diver fell from an altitude of 5000 feet, find the sky diver's position at the instant the parachute is released.
SOLUTION: Solve for C using -5000 (down is positive in our model):

$$s(t) = 1800\mathrm{e}^{-(2/15)t} + 240t - 6800$$

so at $t = 10$, the position s is approximately: -3925.52, which is about 3926 feet high.

(c) After the parachute opens, is there a limiting velocity? If so, find it. (HINT: You do not need to re-solve the DE).
SOLUTION: Yes- The equilibrium solution to the model equation (after the parachute is pulled) is 15 feet per second (which we can also see from the solution).

27. Rewrite the following differential equations as an equivalent system of first order equations. If it is an IVP, also determine initial conditions for the system.

(a) $y'' - 3y' + 4y = 0$, $y(0) = 1$, $y'(0) = 2$.
SOLUTION: Let $x_1 = y$ and $x_2 = y'$:

$$\begin{array}{rcl} x_1' &=& x_2 \\ x_2' &=& -4x_1 + 3x_2 \end{array} \quad \Rightarrow \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

(b) $y''' - 2y'' - y' + 4y = 0$

SOLUTION: Same idea as before, but now we need three variables. Let $x_1 = y$, $x_2 = y'$ and $x_3 = y''$. Then:

$$\begin{array}{rlll} x_1' & = & x_2 & \\ x_2' & = & & x_3 \\ x_3' & = -4x_1 & x_2 & +2x_3 \end{array}$$

(c) $y'' - yy' + t^2 = 0$

SOLUTION: This equation is nonlinear and non-autonomous, but the substitutions work just as before: Let $x_1 = y, x_2 = y'$. Then:

$$\begin{array}{rl} x_1' & = x_2 \\ x_2' & = x_1x_2 - t^2 \end{array}$$

28. Convert one of the variables in the following systems to an equivalent higher order differential equation, and solve it (be sure to solve for both x and y):

$$\begin{array}{rl} x' & = 4x + y \\ y' & = -2x + y \end{array}$$

SOLUTION: Solve the first equation for y, substitute into the second equation to get:

$$x'' - 5x' + 6x = 0 \quad \Rightarrow \quad (r-3)(r-2) = 0$$

Therefore, $x(t) = C_1\mathrm{e}^{3t} + C_2\mathrm{e}^{2t}$. To find y, we have to compute $x' - 4x$, which is:

$$y(t) = 3C_1\mathrm{e}^{3t} + 2C_1\mathrm{e}^{2t} - 4(C_1\mathrm{e}^{3t} + C_2\mathrm{e}^{2t})$$

Therefore, putting these together as a parametric solution:

$$\left[\begin{array}{c} x \\ y \end{array}\right] = C_1\mathrm{e}^{3t}\left[\begin{array}{r} 1 \\ -1 \end{array}\right] + C_2\mathrm{e}^{2t}\left[\begin{array}{r} 1 \\ -2 \end{array}\right]$$

29. Solve the previous system by using eigenvalues and eigenvectors.

SOLUTION: As a hint, we should be able to read the eigenvalues and eigenvectors off of the given solution, but we'll go through it the long way just to be double check the method.

$$A = \left[\begin{array}{rr} 4 & 1 \\ -2 & 1 \end{array}\right] \qquad \begin{array}{rl} \mathrm{Tr}(A) & = 5 \\ \det(A) & = 6 \end{array} \qquad \Rightarrow \quad \lambda^2 - 5\lambda + 6 = 0$$

And this is indeed the same characteristic function as before. We'll work with $\lambda = 3$ first, and:

$$\begin{array}{rl} (4-3)v_1 + v_2 & = 0 \\ -2v_1 + (1-3)v_2 & = 0 \end{array} \quad \Rightarrow \quad \begin{array}{rl} v_1 & = v_1 \\ v_2 & = -v_1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{r} 1 \\ -1 \end{array}\right]$$

A similar computation shows that $[1, -2]^T$ is an eigenvector for $\lambda = 2$. Therefore, we get the same general solution as before.

30. Verify by direct substitution that the given power series is a solution of the differential equation:

$$y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$(x+1)y'' + y' = 0$$

SOLUTION: If you get stuck on this one, you might think about what this function is without the sum:

$$y = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \frac{1}{6}x^6 + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

Therefore,

$$y' = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

$$y'' = -1 + 2x - 3x^2 + 4x^3 - 5x^4 + \cdots = \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2}$$

So you can check that $(1+x)y'' + y' = 0$ for these terms- However, we should be able to show it in general using the sum notation.

We need to show that the following polynomial is zero for all x:

$$(1+x)\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

which is easiest if we have a single sum:

$$\sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-1} + \sum_{n=2}^{\infty}(-1)^{n+1}(n-1)x^{n-2} + \sum_{n=1}^{\infty}(-1)^{n+1}x^{n-1}$$

Combining to have x^k with $k = 0, 1, 2, \cdots$, we have:

$$\sum_{k=0}^{\infty}\left(k(-1)^{k+2} + (k+1)(-1)^{k+3} + (-1)^{k+2}\right)x^k$$

The coefficients become:

$$(k+1)(-1)^{k+2} + (k+1)(-1)^{k+3}$$

so, if k is even, the expression is 0 and if k is odd the expression is zero- Every coefficient is zero.

31. Convert the given expression into a single power series:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^n + 2\sum_{n=2}^{\infty} n a_n x^{n-2} + 3\sum_{n=1}^{\infty} a_n x^n$$

32. Find the recurrence relation for the coefficients of the power series solution to $y'' - (1+x)y = 0$ at $x_0 = 0$.

SOLUTION: Substitute the power series into the DE to get:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - (1+x)\sum_{n=0}^{\infty} a_n x^n = 0$$

Break up the second sum:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

We want to collect these into a single sum- Looks like we might need to peel the constant terms off of the first and second sums to get all series to start with the first power:

$$2 \cdot 1 \cdot a_2 + \sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} - \left(a_0 + \sum_{n=1}^{\infty} a_n x^n\right) - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now we can write everything as a single sum:

$$(2a_2 - a_0) + \sum_{k=1}^{\infty}\left((k+2)(k+1)a_{k+2} - a_k - a_{k-1}\right)x^k = 0$$

from which we get the recurrence relation:

$$a_2 = \frac{1}{2}a_0 \quad \text{and } a_{k+2} = \frac{a_k + a_{k-1}}{(k+2)(k+1)} \quad \text{for } k = 1, 2, 3, \cdots$$

33. Find the first 5 non-zero terms of the series solution to $y'' - (1+x)y = 0$ if $y(0) = 1$ and $y'(0) = -1$ (use derivatives).

SOLUTION: Taking derivatives,

$$y'' = (1+x)y \qquad y^{(3)} = y + (1+x)y' \qquad y^{(4)} = 2y' + (1+x)y''$$

and so on. From this we see that (remember to divide by the appropriate factorial):

$$y(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{40}x^5 + \cdots$$

EXTRA: You might notice that the derivatives have the relation:

$$y^{(k+2)}(0) = ky^{(k-1)}(0) + y^{(k)}(0)$$

Substituting from the Taylor formula $\frac{y^{(k)}}{k!} = a_k$, or $y^{(k)} = k!a_k$ we get:

$$(k+2)!a_{k+2} = k(k-1)!a_{k-1} + k!a_k$$

Simplifying, we get

$$a_{k+2} = \frac{a_k + a_{k-1}}{(k+1)(k+2)}$$

which you might recognize from the previous problem!

34. We have two tanks, A and B with 20 and 30 gallons of fluid, respectively. Water is being pumped into Tank A at a rate of 2 gallons per minute, 2 ounces of salt per gallon. The well-mixed solution is pumped out of Tank A and into Tank B at a rate of 4 gallons per minute. Solution from Tank B is entering Tank A at a rate of 2 gallons per minute. Water is being pumped into Tank B at k gallons per minute with 3 ounces of salt per gallon. The solution is being pumped out of tank B at a total rate of 5 gallons per minute (2 of them are going into tank A).

- What should k be in order for the amount of solution in Tank B to remain at 30? Use this value for the remaining problems.
  $k = 1$
- Write the system of differential equations for the amount of salt in Tanks A, B at time t. Do not solve.
  Let $A(t), B(t)$ be the amount (in ounces) of salt in Tanks A, B respectively. Then:

$$\frac{dA}{dt} = 4 + \frac{2}{30}B - \frac{4}{20}A$$

$$\frac{dB}{dt} = 3 + \frac{4}{20}A - \frac{5}{30}B$$

- Find the equilibrium solution and classify it.
  Solve:

$$\begin{array}{rl} 4 - \frac{1}{5}A + \frac{1}{15}B & = 0 \\ 3 + \frac{1}{5}A - \frac{1}{6}B & = 0 \end{array} \quad \Rightarrow \quad A = \frac{130}{3} \quad B = 70$$

  Notice that these quantities are in total ounces in each tank.
  We can classify this equilibrium by looking at the matrix associated with the system:

$$\begin{bmatrix} -\frac{1}{5} & \frac{1}{15} \\ \frac{1}{5} & -\frac{1}{6} \end{bmatrix}$$

  The trace is negative, the determinant is positive, and the discriminant is positive. The equilibrium is a SINK (which is what we would expect from the physical problem).

35. Solve, and determine how the solution depends on the initial condition, $y(0) = y_0$: $y' = 2ty^2$

$$y(t) = \frac{-y_0}{y_0 t^2 - 1}$$

If $y_0 > 0$, then the solution will only be valid between $\pm\frac{1}{\sqrt{y_0}}$. If $y_0 < 0$, the solution will be valid for all t.

36. Solve the linear system $\mathbf{x}' = A\mathbf{x}$ using eigenvalues and eigenvectors, if A is as defined below:

(a) $A = \begin{bmatrix} 2 & 8 \\ -1 & -2 \end{bmatrix}$ SOLN: $\lambda = \pm 2i$

For $\lambda = 2i$, the equation $(A - \lambda I)\mathbf{v} = 0$ simplifies to:

$$\begin{aligned} v_1 &= (-2-2i)v_2 \\ v_2 &= v_2 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} -2-2i \\ 1 \end{bmatrix}$$

Now we compute $\mathrm{e}^{\lambda t}\mathbf{v}$ before writing the solution:

$$(\cos(2t) + i\sin(2t)) \begin{bmatrix} -2-2i \\ 1 \end{bmatrix} = \begin{bmatrix} (-2\cos(2t) + 2\sin(2t)) + i(-2\cos(2t) - 2\sin(2t)) \\ \cos(2t) + i\sin(2t) \end{bmatrix}$$

The solution to the DE is now:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} -2\cos(2t) + 2\sin(2t) \\ \cos(2t) \end{bmatrix} + C_2 \begin{bmatrix} -2\cos(2t) - 2\sin(2t) \\ \sin(2t) \end{bmatrix}$$

(b) $A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$ SOLN: $\lambda^2 - 3\lambda - 4 = 0$, so $\lambda = -1, 4$.

If $\lambda = -1$, then $v_1 + v_2 = 0$, or

$$\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

If $\lambda = 4$, then $2v_1 + 3v_2 = 0$, or

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Therefore, the general solution is:

$$\mathbf{x}(t) = C_1 \mathrm{e}^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \mathrm{e}^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

(c) $A = \begin{bmatrix} 3 & -18 \\ 2 & -9 \end{bmatrix}$

You should see that $\lambda = -3, -3$. Putting $\lambda = -3$ into the equation for the eigenvector, we get:

$$\begin{aligned} 6v_1 - 18v_2 &= 0 \\ 2v_1 - 6v_2 &= 0 \end{aligned} \quad \Rightarrow \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Computing the generalized eigenvector $\mathbf{w}$, we get:

$$\begin{aligned} 6w_1 - 18w_2 &= 3 \\ 2w_1 - 6w_2 &= 1 \end{aligned} \quad \Rightarrow \quad \mathbf{w} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

where we choose one convenient solution for $\mathbf{w}$. The general solution is then:

$$\mathbf{x}(t) = \mathrm{e}^{-3t} \left[C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 \left(t \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right) \right]$$

37. For each nonlinear system, find and classify the equilibria:

(a) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1+2y \\ 1-3x^2 \end{bmatrix}$

SOLUTION: The equilibria are at $y = -\frac{1}{2}$ and $x = \pm\frac{1}{\sqrt{3}}$. The Jacobian matrix is:

$$\begin{bmatrix} 0 & 2 \\ -6x & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathrm{Tr}(A) &= 0 \\ \det(A) &= 12x \\ \Delta &= -48x \end{aligned}$$

Therefore, there is a CENTER at $(1/\sqrt{3}, -1/2)$ and a SADDLE at $(-1/\sqrt{3}, 1/2)$.

(b) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1-y \\ x^2-y^2 \end{bmatrix}$

SOLUTION: The equilibria are at $(1,1)$ and $(-1,1)$. The Jacobian matrix is:

$$\begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathrm{Tr}(A) &= -2y \\ \det(A) &= 2x \\ \Delta &= 4y^2 - 8x \end{aligned}$$

There is a spiral sink at $(1,1)$ and a saddle at $(-1,1)$.

(c) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x+x^2+y^2 \\ y(1-x) \end{bmatrix}$

SOLUTION: From the second equation, $y=0$ or $x=1$. If $y=0$, then from the first equation, $x^2+x=0$, or $x(x+1)=0$, so $x=0$ or $x=-1$. If $x=1$ from the second equation, then in the first equation, $y^2+2=0$ which has no real solution. Therefore, the equilibria are $(0,0)$ and $(-1,0)$. The Jacobian matrix is:

$$\begin{bmatrix} 1+2x & 2y \\ -y & 1-x \end{bmatrix} \quad \Rightarrow \quad J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad J(-1,0) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

There is a source at the origin (OK if you say it is degenerate, since that is what is on the diagram). The other equilibrium $(-1,0)$ is a saddle. (This one is kind of fun to plot!)

(d) $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x-y^2 \\ y-x^2 \end{bmatrix}$

The equilibria are at $(0,0)$ and $(1,1)$. The Jacobian matrix is:

$$\begin{bmatrix} 0 & -2y \\ -2x & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} \mathrm{Tr}(A) &= 2 \\ \det(A) &= 1-4xy \\ \Delta &= 16xy \end{aligned}$$

At $(0,0)$, there is a source (OK to say degenerate source), and at $(1,1)$ there is a saddle.

Side Remark: If the Jacobian matrix is especially simple, sometimes we can save time evaluating it by computing the trace, determinant and discriminant in terms of x and y. If the Jacobian is at all complicated, then it's probably easier to evaluate the matrices first (like in (c)).